Euler and Lagrange descriptions

Euler approach The fluid properties $p, \rho, v$ are written as functions of space and times. The flow is determined by the analyzing the behavior of the functions.

Lagrange approach Pieces of the fluid are “tagged”. The fluid flow properties are determined by tracking the motion and properties of the particles as they move in time.
Euler vs Lagrange

Consider smoke going up a chimney

**Euler approach** Attach thermometer to the top of chimney, point 0. Record $T$ as a function of time. As different smoke particles pass through $O$, the temperature changes. Gives $T(x_0, y_0, z_0, t)$. More thermometers to get $T(x, y, z, t)$.

**Lagrange approach** Thermometers are attached to a particle, $A$. End up with $T_A = T_A(a)$. Can have many particles and track $T$ for all of them. If we also know, position of each particle of function of time, can translate Lagrange information into Euler information.
Euler vs Lagrange

It is generally more common to use Eulerian approach to fluid flows. Measuring water temperature, or pressure at a point in a pipe. Lagrangian methods sometimes used in experiments. Throwing tracers into moving water bodies to determine currents (see movie Twister). X-ray opaque tracers in human blood.

Bird migration example. Ornithologists with binoculars count migrating birds moving past a (Euler) or scientists place radio transmitters on the birds (Lagrange).
Streamlines, streaklines, pathlines

Streamlines, streaklines and pathlines are used in the visualization of fluid flow. Streamlines mainly used in analytic work, streaklines and pathlines used in experimental work.

Streamlines are tangent to the velocity field. For steady flow, the streamlines are fixed in space. Unsteady flow, streamlines may change with time. The slope of the streamline is equal to tangent of velocity field.

\[
\frac{dy}{dx} = \frac{v}{u}
\]

The streamlines can be determined from velocity field by integrating the lines define the tangents.
**Streaklines**

**streak-lines** Consist of all the particles in a flow that have passed through a common point. Mainly a laboratory tool.

A streak-line can be made by injecting dye into a moving fluid at a specific point. For a steady flow, each particle follows the previous ones precisely, and the streak-line is the same as the streamline.

For unsteady flows, particles injected at the same point at different times need not follow the same path. An instantaneous photograph of the marked fluid would show the streak. The streak-line would not be the same as the streamline.

**Pathline** This is the trajectory followed by *one* particle when it moves from one point to the next. One injects dye at a point for an instant of time, then does a time exposure photograph.

For steady flow *streamline = streakline = pathline*
Streaklines

Streaklines of dye moving past obstruction. They are also the streamlines for the flow.

Streaklines of smoke moving past obstruction.
The material derivative

In the Eulerian method, the fundamental property is the velocity field. The velocity field does not track the behaviour of individual partials, it describes the velocity of whatever happens to be at a given location. To do dynamics, need to apply $F = ma$. Getting the acceleration is not trivial

For particle $A$, $x_A(t), y_A(t), z_A(t)$ describe the motion of the particle. So

$$v_A = v_A(x_A(t), y_A(t), z_A(t), t)$$
The material derivative

\[ a_A = \frac{d\mathbf{v}_A}{dt} \]

\[ = \frac{\partial \mathbf{v}_A}{\partial t} + \frac{\partial \mathbf{v}_A}{\partial x} \frac{d\mathbf{x}_A}{dt} + \frac{\partial \mathbf{v}_A}{\partial y} \frac{d\mathbf{y}_A}{dt} + \frac{\partial \mathbf{v}_A}{\partial z} \frac{d\mathbf{z}_A}{dt} \]

\[ = \frac{\partial \mathbf{v}_A}{\partial t} + \frac{\partial \mathbf{v}_A}{\partial x} u_A(t) + \frac{\partial \mathbf{v}_A}{\partial y} v_A(t) + \frac{\partial \mathbf{v}_A}{\partial z} w_A(t) \]

Since \( A \) is any particle, the acceleration field is

\[ \mathbf{a} = \frac{\partial \mathbf{v}}{\partial t} + u \frac{\partial \mathbf{v}}{\partial x} + v \frac{\partial \mathbf{v}}{\partial y} + w \frac{\partial \mathbf{v}}{\partial z} \]

The \( \mathbf{a} \) is a vector with components

\[ a_x = \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \]

\[ a_y = \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \]

\[ a_z = \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \]
The material derivative

The material derivative is often written

\[ \mathbf{a} = \frac{D\mathbf{v}}{Dt} \]

with

\[ \frac{D()}{Dt} = \frac{\partial()}{\partial t} + u \frac{\partial()}{\partial x} + v \frac{\partial()}{\partial y} + w \frac{\partial()}{\partial z} \]

One can define the material derivative for other properties for a fluid, e.g. temperature or pressure.

\[ \frac{DT}{Dt} = \frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} + w \frac{\partial T}{\partial z} \]

The material derivative allows for two types of contribution. Unsteady effects when \( \frac{\partial}{\partial t} \neq 0 \) and convective when \( \frac{\partial}{\partial xyz} \neq 0 \).
**Unsteady effects**

Consider water from a header tank flowing down a uniform cross section pipe.

The water velocity at all points will be the same. However the water velocity will gradually decrease as the header tank empties.

\[
a = \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z}
\]

\[
a = \frac{\partial v}{\partial t} + 0 + 0 + 0
\]

The only term to survive is the *local acceleration*, namely \(\frac{\partial v}{\partial t}\). The \(\frac{\partial}{\partial t}\) part of the material derivative is called the local derivative.
Convective derivative

Consider water going through a water heater under steady state flow conditions.

The water temperature at any fixed location is fixed, i.e. $\frac{\partial T}{\partial t} = 0$.

However, the water temperature for a given piece of water will increase as it progresses through the heater. The rate of change is

$$\frac{dT}{dt} = \left( \begin{array}{c} \text{Rate at which } T \text{ changes} \\ \text{with position} \end{array} \right) \times \left( \begin{array}{c} \text{How quickly} \\ \text{water changes} \\ \text{position} \end{array} \right)$$

$$= \frac{\partial T}{\partial s} u_s$$
Convective derivative

The *convective part* of the material derivative

\[
\frac{D()}{Dt} = u \frac{\partial()}{\partial x} + v \frac{\partial()}{\partial y} + w \frac{\partial()}{\partial z}
\]

represents changes in the flow properties associated with the movement of a particle from one point in space to another. So movement to another location can also affect the net time rate of change of small pieces of the fluid.