IL NUOVO CIMENTO VOL. 14 B, N. 2 11 Aprile 1973

Galilean Electromagnetism.

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(ricevuto il 6 Novembre 1972)

Summary. — Consistent nonrelativistic electromagnetic theories are investigated by stressing the requirements of Galilean relativity. It is shown that Maxwell's equations admit two possible nonrelativistic limits, accounting respectively for electric and magnetic effects. A Galilean theory is then built which combines these two theories and can embody a large class of experimental facts. As a result, several so-called «relativistic» effects are shown to necessitate a re-appraisal, or at least, a more careful discussion. It is finally shown precisely how the old-fashioned formulation of the electromagnetic theory in terms of field strengths and field excitations clashes with Galilean relativity in its constitutive equations only, leading to the idea of a privileged frame of reference (the ether) or to Einsteinian relativity!

1. – Introduction.

Does there exist a consistent and physically meaningful nonrelativistic theory of classical electromagnetism? The present paper tries to answer this question, the paradoxical nature of which seems obvious almost seventy years after the advent of relativity theory, forced upon physics by the very difficulties of nonrelativistic electromagnetism. We maintain however and wish

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to convince the reader that the real paradox rather lies in the lack of rigour and the vagueness which characterizes the present understanding of the «relativistic» aspects of electromagnetic theory. This is partly due to the generally imprecise meaning attached to the qualification of a theory or a physical effect as «nonrelativistic». We feel that there is but one consistent definition, according to which «nonrelativistic» is taken to mean «in agreement with the principle of Galilean relativity». Numerous recent developments have shown rather surprising aspects of Galilean physics as studied with the tools and from the points of view developed for (Einsteinian) relativistic physics (1). These studies are interesting because of their direct applications (why use Einsteinian relativity when the (simpler) Galilean one is sufficient?) as well as by the light they shed upon the relativistic theories themselves.

For these reasons we feel it necessary to investigate some questions, such as:

a) What are the Galilean limits—if they exist—of the Maxwell equations and the Lorentz force? As a corollary: which electromagnetic effects may be considered as nonrelativistic, and which can only show up in a relativistic theory?

b) How do the electric and magnetic fields behave under a Galilean transformation? We believe that this point is important, because one can find in some well-known textbooks (2) «low-velocity formulae» which are incorrect, or at least misleading, in the sense that they do not correspond to any consistent Galilean limit.

c) More generally, what equations could have been written in the 1850's by a physicist trying to embody the known phenomena of electromagnetism in a Galilean invariant theory?

It is not trivial to answer the above questions, and, to begin with, let us remark at once that one cannot take carelessly the limit $c\to\infty$ in Maxwell's equations, since the result simply depends upon one's favourite system of units (3)! It turns out, however, that correct results may be obtained by taking very carefully the limit $c\to\infty$, as will be seen later (see the Conclusions).

However, we will not rely on such a limit process to establish our results. At this point, let us only notice that the CGS system of units is *a priori* very

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(1) For a general account of Galilean invariance and its applications, see the review article by J.-M. LÉVY-LEBLOND: in *Group Theory and its Applications*, Vol. **2**, edited by E. M. LOEBL (New York, 1971), where references to the original literature will be found.

(2) E. M. PURCELL: *Berkeley Physics Course*, Vol. **2**, *Electricity and Magnetism*, Problem 6-10 (Reading, Mass, 1965); L. D. LANDAU and E. M. LIFSCHITZ: *The Classical Theory of Fields*, Sect. **3-10** (New York, 1959).

(3) Systems of units in electromagnetism are beautifully discussed by A. SOMMERFELD: *Electrodynamics*, Sect. **2**, **7** and **8** (New York, 1952).

inconvenient for our purposes, since the velocity of light enters its very definition. We shall thus use a system of units of the MKSA type, depending on two constants ε_0 and μ_0, where ε_0 (μ_0) is defined by the force between two charges (currents) and the unit of charge (current) in the usual way.

Our main observation is the following: there exist two different and perfectly well-defined Galilean limits of classical electromagnetism, and not just a single « nonrelativistic » limit. The first limit is valid when electric effects are dominant ($E \gg cB$): we call it the *electric limit*; the second one holds when magnetic effects are dominant ($cB \gg E$): it will be called the *magnetic limit*. Although these limits remain in any case the fundamental ones, they can be combined in order to build more general Galilean invariant theories, which will be described in Sect. **3** and Appendix B.

From a mathematical point of view, the existence of two limits arises because there are two different kinds of « Galilean four-vectors », and thus two possible descriptions of a Galilean current four-vector. Both are physically relevant because there exist positive and negative electric charges: this is why there are physical situations where either electric effects, or magnetic effects, are dominant.

As a consequence, it is not so easy to decide whether a given phenomenon is relativistic or not. For example, the magnetic force between two currents is certainly a relativistic effect, if the current is considered as a transport of charge. However, in the framework of the magnetic limit, it is perfectly possible to write down a phenomenological Galilean invariant theory of magnetism and magnetic forces. The plan of the paper is as follows. In Sect. **2** we derive the equations of the two fundamental limits. In Sect. **3** we build a general Galilean theory of electromagnetism, combining, so to speak, these two Galilean limits, by introducing two kinds of electromagnetic fields, and we discuss its physical interpretation. Finally, in Sect. **4**, we examine the problem from an entirely different point of view, by working with the electric and magnetic excitations $\boldsymbol{D}$ and $\boldsymbol{H}$ in addition to $\boldsymbol{E}$ and $\boldsymbol{B}$: we show how the breaking of Galilean invariance can be pushed into the constitutive equations $\boldsymbol{D} = \varepsilon\boldsymbol{E}$ and $\boldsymbol{B} = \mu\boldsymbol{H}$, which lead naturally to the hypothesis of an absolute frame of reference. Appendix A connects our Galilean electromagnetic theories with the general theory of Galilean wave equations, Appendix B exhibits two « improved » Galilean limits of the Maxwell's equations, Appendix C deals with the delicate problem of determining if the spin-orbit coupling exists in a nonrelativistic theory (and how much of it).

2. – The two fundamental limits.

2'1. *Galilean transformation for the current four-vector.* – That two independent limits are necessary can be seen most clearly by working out the Galilean transformation law for the current four-vector $(c\varrho, \boldsymbol{j})$. Indeed, the usual Lorentz transformation of a four-vector $(u_0, \boldsymbol{u})$ admits two different Galilean limits, *viz.*

$$(2.1)\qquad \begin{cases} u_0' = u_0\,, \\ \boldsymbol{u}' = \boldsymbol{u} - \dfrac{1}{c}\,\boldsymbol{v}u_0\,, \end{cases}$$

$$(2.2)\qquad \begin{cases} u_0' = u_0 - \dfrac{1}{c}\,\boldsymbol{v}\cdot\boldsymbol{u}\,, \\ \boldsymbol{u}' = \boldsymbol{u}\,, \end{cases}$$

where $\boldsymbol{v}$ is the velocity of the transformation.

Equations (2.1) are valid if $v/c \ll 1$ and $|\boldsymbol{u}| \ll |u_0|$, *i.e.* if the four-vector is « largely timelike »; for example the usual Galilean transformation

$$(2.3)\qquad \begin{cases} c\Delta t' = c\Delta t\,, \\ \Delta\boldsymbol{r}' = \Delta\boldsymbol{r} - \boldsymbol{v}\Delta t\,, \end{cases}$$

only holds if $|\Delta\boldsymbol{r}| \ll c|\Delta t|$. Observe that the spatio-temporal gradient obeys the alternate transformation law

$$(2.4)\qquad \begin{cases} \dfrac{1}{c}\dfrac{\partial}{\partial t'} = \dfrac{1}{c}\dfrac{\partial}{\partial t} - \dfrac{1}{c}\,\boldsymbol{v}\cdot\boldsymbol{\nabla}\,, \\ \boldsymbol{\nabla}' = \boldsymbol{\nabla}\,. \end{cases}$$

Indeed eqs. (2.2) must be used if $v/c \ll 1$ and $|\boldsymbol{u}| \gg |u_0|$, that is, for a « largely spacelike » four-vector. Applying eqs. (2.1) and (2.2) to the current four-vector, we obtain what will be seen to be the electric limit (eqs. (2.1)) if $c|\varrho| \gg |\boldsymbol{j}|$ and the magnetic limit (eqs. (2.2)) if $c|\varrho| \ll |\boldsymbol{j}|$. Obviously, if there existed only positive (or negative) electric charges, the electric limit alone would be physically relevant. The existence of two types of electric charges allows $|\boldsymbol{j}|$ to be much larger than $c|\varrho|$ in many cases—this is the usual situation at a macroscopic level (in particular when $\varrho = 0$ while $\boldsymbol{j} \neq 0$)—so that both limits are physically interesting.

2'2. *Electric limit.* – Let us first turn to the case $c|\varrho| \gg |\boldsymbol{j}|$, which implies that $E \gg cB$. This is the electric limit (subscript e) which corresponds to the transformation law (2.1) for $(c\varrho, \boldsymbol{j})$:

$$(2.5)\qquad \begin{cases} \varrho_e' = \varrho_e\,, \\ \boldsymbol{j}_e' = \boldsymbol{j}_e - \boldsymbol{v}\varrho_e\,. \end{cases}$$

Notice that the continuity equation

$$\frac{\partial \varrho_e}{\partial t} + \nabla \cdot \boldsymbol{j}_e = 0 \tag{2.6}$$

is Galilean invariant according to eqs. (2.4) and (2.5) and continues to hold in this limit: the current $\boldsymbol{j}_e$ is a transport of charge. Taking into account that $|\boldsymbol{E}_e| \gg c|\boldsymbol{B}_e|$, we can derive the following transformation law for the electromagnetic field (from its usual Lorentz transformation):

$$\begin{cases} \boldsymbol{E}'_e = \boldsymbol{E}_e\,, \\ \boldsymbol{B}'_e = \boldsymbol{B}_e - \varepsilon_0\mu_0 \boldsymbol{v} \times \boldsymbol{E}_e\,. \end{cases} \tag{2.7}$$

These last equations show that the motion of an electric field (and, more generally, any time variation) induces a magnetic field, while a time-varying magnetic field does not induce any electric field. Thus Faraday's law of induction is no longer true in this limit: there can be no « Faraday term » $-\partial \boldsymbol{B}_e/\partial t$ in Maxwell's equations, which now read

$$\begin{cases} \nabla \cdot \boldsymbol{E}_e = \varrho_e/\varepsilon_0\,, \\ \nabla \times \boldsymbol{E}_e = 0\,, \end{cases} \qquad \begin{cases} \nabla \cdot \boldsymbol{B}_e = 0\,, \\ \nabla \times \boldsymbol{B}_e = \varepsilon_0\mu_0 \dfrac{\partial \boldsymbol{E}_e}{\partial t} + \mu_0 \boldsymbol{j}_e\,. \end{cases} \tag{2.8}$$

By using eqs. (2.4) it is straightforward to check that eqs. (2.8) are invariant under the Galilean transformations (2.5)-(2.7). Physically, the theory based on these equations will describe situations where isolated electric charges move with low velocities. The electric field $\boldsymbol{E}_e$ is derived from a scalar potential φ_e, the magnetic field $\boldsymbol{B}_e$ from a vector potential $\boldsymbol{A}_e$:

$$\begin{cases} \boldsymbol{E}_e = -\nabla\varphi_e\,, \\ \boldsymbol{B}_e = \nabla \times \boldsymbol{A}_e\,, \end{cases} \tag{2.9}$$

which obey the transformation law

$$\begin{cases} \varphi'_e = \varphi_e\,, \\ \boldsymbol{A}'_e = \boldsymbol{A}_e - \varepsilon_0\mu_0 \boldsymbol{v}\varphi_e\,. \end{cases} \tag{2.10}$$

Hence the 4-potential $(\varphi_e, c\boldsymbol{A}_e)$ is a 4-vector of the type (2.1). Now, eqs. (2.8) do not yet determine the theory, since we have not given the limit of the Lorentz force

$$\boldsymbol{F} = \int \mathrm{d}^3 r\, [\varrho(\boldsymbol{r})\boldsymbol{E}(\boldsymbol{r}) + \boldsymbol{j}(\boldsymbol{r}) \times \boldsymbol{B}(\boldsymbol{r})]\,. \tag{2.11}$$

If one demands Galilean invariance of the force ($F' = F$), it is easy to convince oneself that a magnetic force $\int d^3r\, \boldsymbol{j}_e \times \boldsymbol{B}_e$ is inconsistent with the field transformation law (2.7). Hence we can only have electric forces given by

$$F_e = \int d^3r\, \varrho_e(\boldsymbol{r}) \boldsymbol{E}_e(\boldsymbol{r})\,. \tag{2.12}$$

This, indeed, is the limit of the expression (2.11) under the « electric » conditions $c|\varrho| \gg |\boldsymbol{j}|$ and $|\boldsymbol{E}| \gg c|\boldsymbol{B}|$. Thus, in the electric limit, the magnetic field $\boldsymbol{B}_e$ does exist, but has no effect at all! However, in the more general theory developed in Sect. **3**, there will be magnetic forces due to $\boldsymbol{B}_e$.

2·3. *Magnetic limit.* – We now consider the opposite case, in which $c|\varrho| \ll \ll |\boldsymbol{j}|$ or $|\boldsymbol{E}| \ll c|\boldsymbol{B}|$ in order to obtain the magnetic limit (subscript m). This limit provides a phenomenological theory of magnetostatics, and may be applied to the usual situations at a macroscopic level, where magnetic effects are in general dominant because of the balance between positive and negative charges. In this limit, the current four-vector transforms according to eqs. (2.2):

$$\begin{cases} \varrho'_m = \varrho_m - \varepsilon_0 \mu_0 \boldsymbol{v} \cdot \boldsymbol{j}_m\,, \\ \boldsymbol{j}'_m = \boldsymbol{j}_m\,. \end{cases} \tag{2.13}$$

Taking into account that $|\boldsymbol{E}_m| \ll c|\boldsymbol{B}_m|$, one obtains the field transformation law

$$\begin{cases} \boldsymbol{E}'_m = \boldsymbol{E}_m + \boldsymbol{v} \times \boldsymbol{B}_m\,, \\ \boldsymbol{B}'_m = \boldsymbol{B}_m\,. \end{cases} \tag{2.14}$$

These equations imply that the motion of a magnetic field (or its time variation) induces an electric field, while a time-varying electric field does not produce any magnetic field. Hence there can be no displacement current in Maxwell's equations which read in this limit

$$\begin{cases} \nabla \cdot \boldsymbol{E}_m = \varrho_m / \varepsilon_0\,, \\ \nabla \times \boldsymbol{E}_m = -\dfrac{\partial \boldsymbol{B}_m}{\partial t}\,, \end{cases} \qquad \begin{cases} \nabla \cdot \boldsymbol{B}_m = 0\,, \\ \nabla \times \boldsymbol{B}_m = \mu_0 \boldsymbol{j}_m\,. \end{cases} \tag{2.15}$$

Again, it is straightforward to check that eqs. (2.15) are invariant under the Galilean transformations (2.13) and (2.14). It is very important to remark that eqs. (2.13) (or (2.15)) *allow only stationary currents*

$$\nabla \cdot \boldsymbol{j}_m = 0\,, \tag{2.16}$$

and the current $\boldsymbol{j}_m$ cannot be related to a transport of charge.

In this limit, there cannot be any accumulation of charge in a fixed volume. The fields E_m and B_m may be derived from potentials φ_m and A_m according to

$$(2.17)\qquad \begin{cases} E_m = -\nabla\varphi_m - \dfrac{\partial A_m}{\partial t}, \\ B_m = \nabla\times A_m, \end{cases}$$

and the 4-potential (φ_m, cA_m) is in this case a 4-vector of the type (2.2):

$$(2.18)\qquad \begin{cases} \varphi'_m = \varphi_m - v\cdot A_m, \\ A'_m = A_m. \end{cases}$$

As in the electric limit, we must finally give the force law; it is easy to see that an electric force

$$\int d^3r\, \varrho_m(r)\, E_m(r)$$

is inconsistent with Galilean invariance, and there can exist only magnetic forces:

$$(2.19)\qquad F_m = \int d^3r\, j_m(r)\times B_m(r).$$

This is also the correct form of expression (2.11) in the magnetic limit $c|\varrho| \ll |j|$ and $|E| \ll c|B|$. Hence the electric field E_m is nonzero, but it does not produce any observable effect.

2·4. *Remarks.* – i) Equations (2.8) contain the displacement current, and eqs. (2.15) Faraday's law of induction. However the induced fields B_e and E_m have a rather formal existence, since they do not produce any effect. In the next Section we shall build a more complete theory, where B_e and E_m do produce observable effects.

ii) The total charge ΔQ_m induced by a change of frame of reference vanishes; indeed, following (2.13), we have

$$(2.20)\qquad \Delta Q_m = \int \Delta\varrho_m(r)\, d^3r = -\varepsilon_0\mu_0 \int d^3r\, v\cdot j_m(r).$$

By remembering that $\mu_0 j_m = \nabla\times B_m$ (see (2.15)) one transforms ΔQ_m in a vanishing surface integral

$$(2.21)\qquad \Delta Q_m = \varepsilon_0 v\cdot \int B_m \times d^2 s = 0.$$

This is a typically Galilean situation where there is a global conservation law (Q_m = const) without a local equivalent, *i.e.* ϱ_m and $\boldsymbol{j}_m$ do not obey a continuity equation. The variation of charge in a given volume is not due to a current flow through the boundary; this agrees of course with our earlier statement about the current $\boldsymbol{j}_m$ not to be thought of as a transport of charges. This may explain how the requirement of *local* charge conservation by MAXWELL, through the introduction of the displacement current, gave rise to a relativistic theory. Indeed, because of the relativity of simultaneity, Einsteinian relativity requires any conservation law to be local. Galilean relativity on the other hand allows for nonlocal conservation laws, with opposite charges for instance appearing at distant points simultaneously, which is a Galilean, but not Einsteinian, invariant statement.

iii) It may seem somewhat strange that the constant μ_0 (ε_0) appears in the electric (magnetic) limit, since this constant cannot be experimentally defined. But one is free to measure ϱ_m in units different from ϱ_e, and $\boldsymbol{B}_e$ in units different from $\boldsymbol{B}_m$. Defining

$$\tilde{\boldsymbol{B}}_e = \boldsymbol{B}_e/\varepsilon_0\mu_0\,, \qquad \tilde{\varrho}_m = \varrho_m/\varepsilon_0\mu_0\,, \tag{2.22}$$

one obtains equations for the electric (magnetic) limit which depend now only on ε_0 (μ_0).

iv) One can find in some textbooks [2] the following « low-velocity limit » for the field transformation law:

$$\begin{cases} \boldsymbol{E}' = \boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B}\,, \\ \boldsymbol{B}' = \boldsymbol{B} - \varepsilon_0\mu_0\boldsymbol{v}\times\boldsymbol{E}\,. \end{cases} \tag{2.23}$$

These equations coincide neither with (2.7) nor with (2.14) and do not correspond to any kind of Galilean limit. Indeed, eqs. (2.23) do not even define a group law: composition of two such transformations does not yield a transformation of the same type. In particular, it is not consistent with the Galilean additivity of velocities. We think that eqs. (2.23) have no well-defined meaning, and should be avoided altogether.

v) We have already pointed out the main defect of both limits, if we are to write down a physically interesting theory of electromagnetism; the fields $\boldsymbol{B}_e$ and $\boldsymbol{E}_m$ produce no effect at all, so that we have, for example, no magnetic force between a current and a moving charge, and no induced current in the presence of time-varying magnetic fields.

vi) Both limits may be viewed as illustrations of the general theory of Galilean wave equations for zero-mass, spin-1 particles (see Appendix A).

3. – A general formulation of Galilean electromagnetism.

3'1. *Definition of the theory.* – The difficulties which were pointed out at the end of the previous Section arise because we require the Galilean invariance of the force. The only way to avoid this difficulty is to distinguish carefully between electric and magnetic charges and currents in their interaction with the fields. It is possible to generalize the previous theories slightly by introducing only one kind of electromagnetic field $(\boldsymbol{E}, \boldsymbol{B})$ (see Appendix B). But in the most general formulation we must consider simultaneously electromagnetic fields of the electric type $(\boldsymbol{E}_e, \boldsymbol{B}_e)$ and of the magnetic type $(\boldsymbol{E}_m, \boldsymbol{B}_m)$. The fields $(\boldsymbol{E}_e, \boldsymbol{B}_e)$ have «electric» sources $(\varrho_e, \boldsymbol{j}_e)$, they obey eqs. (2.8) and have the transformation properties (2.5) and (2.7). Similarly, the fields $(\boldsymbol{E}_m, \boldsymbol{B}_m)$ are generated by «magnetic» sources $(\varrho_m, \boldsymbol{j}_m)$ according to (2.15), with transformation properties given by (2.13) and (2.14). But we now allow $\boldsymbol{B}_e$ to interact with $\boldsymbol{j}_m$ and $\boldsymbol{E}_m$ to interact with ϱ_e according to a force law

$$(3.1)\qquad \boldsymbol{F} = \int \mathrm{d}^3 r\,(\varrho_e \boldsymbol{E}_e + \boldsymbol{j}_m \times \boldsymbol{B}_m + \varrho_e \boldsymbol{E}_m + \boldsymbol{j}_e \times \boldsymbol{B}_m + \varrho_m \boldsymbol{E}_e + \boldsymbol{j}_m \times \boldsymbol{B}_e)\,.$$

In a Galilean transformation, the first two terms in (3.1) are separately invariant, and the same property is also obviously true for the sum of the third and fourth terms. The sum of the fifth and sixth terms is also invariant because, according to (2.7) and (2.13),

$$(3.2)\qquad \boldsymbol{F}' - \boldsymbol{F} = -\varepsilon_0\mu_0 \int \mathrm{d}^3 r\,[(\boldsymbol{v}\cdot\boldsymbol{j}_m)\,\boldsymbol{E}_e + \boldsymbol{j}_m \times (\boldsymbol{v}\times\boldsymbol{E}_e)] =$$
$$= -\varepsilon_0\mu_0 \boldsymbol{v} \int \mathrm{d}^3 r\,(\boldsymbol{j}_m\cdot\boldsymbol{E}_e) = \varepsilon_0\mu_0 \boldsymbol{v} \int \mathrm{d}^3 r\,\nabla\cdot(\varphi_e \boldsymbol{j}_m) = 0\,.$$

Conspicuously missing terms in (3.1) are $\varrho_m \boldsymbol{E}_m$ and $\boldsymbol{j}_e \times \boldsymbol{B}_e$: as was remarked previously, these are precisely the terms which would break the Galilean invariance of the force.

3'2. *Physical interpretation.* – Before we proceed further, we must admit that such a theory looks rather complicated, when compared to the usual one! However we wish to show that it is able to describe a large class of electromagnetic phenomena, and that a physicist deeply concerned about Galilean invariance might have arrived at this formulation in the 1850's.

It is necessary that the electric current in a wire be of the $\boldsymbol{j}_m$-type, otherwise there would be no forces between two currents. Thus we assume that ordinary electric currents—and also magnetization currents—are of the $\boldsymbol{j}_m$-type, while isolated charges are of the ϱ_e-type. The theory is then able to embody the following experimental facts:

i) Existence of forces between isolated charges (Coulomb's law).

ii) Existence of forces between currents and magnets.

iii) Existence of induced currents. More precisely, there exists an induced electric field $\boldsymbol{E}_m$ in the presence of a time-varying magnetic field $\boldsymbol{B}_m$. However, since this field does not act on charges, it is necessary to add a constitutive equation (Ohm's law)

$$\boldsymbol{j}_m = \sigma(\boldsymbol{E}_e + \boldsymbol{E}_m) \tag{3.3}$$

in order to obtain an induced current. Note that despite the transformation laws ((2.7), (2.13), (2.14)), eq. (3.3) does not break Galilean invariance, because it is assumed to hold only in a reference frame where the conducting medium is at rest.

iv) Since $\boldsymbol{B}_e$ acts on $\boldsymbol{j}_m$, moving electric charges exert forces on magnets and currents. This accounts in particular for Rowland's experiment: isolated electric charges in motion may act on a magnet, or a current.

v) For the same reason, there is a spin-orbit coupling: a magnetic moment $\mathcal{M}$ moving with velocity $\boldsymbol{v}$ in an electrostatic field $\boldsymbol{E}_e$ interacts with the magnetic field $\boldsymbol{B}_e = -\varepsilon_0\mu_0 \boldsymbol{v}\times\boldsymbol{E}_e$. Additional comments on this question are offered in Appendix B.

vi) The theory admits nontrivial free fields. The field equations (2.8), (2.15) with null sources imply harmonicity (vanishing Laplacians) of their solution, so that monochromatic « waves » necessarily correspond to uniform, that is constant throughout space, fields. However, there are genuine time-dependent solutions of the following form:

$$\begin{cases} \boldsymbol{E}_e = 0\,, \\ \boldsymbol{B}_e = \boldsymbol{B}_e^{(0)} \exp[i\omega t]\,, \end{cases} \qquad \begin{cases} \boldsymbol{E}_m = \boldsymbol{E}_m^{(0)} \exp[i\omega t]\,, \\ \boldsymbol{B}_m = 0\,, \end{cases} \tag{3.4}$$

where $\boldsymbol{B}_e^{(0)}$ and $\boldsymbol{E}_m^{(0)}$ are uniform fields. Such solutions may be viewed as waves travelling with infinite velocity (since they have an infinite wavelength and a finite frequency). This is quite in keeping with the idea that a Galilean theory cannot admit an intrinsic propagation velocity. These « waves » are able to transmit instantaneous action-at-a-distance since $\boldsymbol{B}_e$ and $\boldsymbol{E}_m$ act on the « true » charges ϱ_e and currents $\boldsymbol{j}_m$ respectively (see eq. (3.1)). Hertz' famous experiments thus might be explained in the present theory, as well as other electromagnetic radiation effects where the finite velocity of propagation is ignored.

Now for the defects: apart from the absence of propagating electromagnetic waves, the most important shortcoming of Galilean electromagnetism is that *capacitors do not work*, in particular with alternating currents! The reason is that the current $\boldsymbol{j}_m$ is necessarily stationary, and there is no continuity equation. Thus there is no relation between the intensity I in the wire and the time derivative $\mathrm{d}Q/\mathrm{d}t$ of the charge stored in the capacitor, and one cannot write down the standard equations. We are forced to conclude that the behaviour of a capacitor is a purely relativistic effect!

Strictly speaking, one could argue that this is not entirely true, since we could have decided the electric currents to be of the $\boldsymbol{j}_e$-type. But then there would be no magnetic force between two currents, and the theory would become rather uninteresting.

As a final remark, let us notice that the so-called quasi-static, or quasi-stationary approximation cannot be made compatible with Galilean invariance.

3·3. *Field energy and field momentum.* – It is not very difficult to obtain the expression for the density of field energy u in this theory. First it is clear that the electrostatic energy is as usual $(\varepsilon_0/2)\boldsymbol{E}_e^2$; the magnetostatic energy can be obtained by using standard methods ([4]), since we had to include Ohm's law (2.3) in our formulation. We thus obtain

$$u=\frac{\varepsilon_0}{2}\boldsymbol{E}_e^2+\frac{1}{2\mu_0}\boldsymbol{B}_m^2\,. \tag{3.5}$$

As usual, we assume that this expression still holds with time-varying fields; Poynting's theorem written in the form

$$\frac{\partial u}{\partial t}+\boldsymbol{E}_e\cdot\boldsymbol{j}_e+\boldsymbol{E}_m\cdot\boldsymbol{j}_m+\nabla\cdot\boldsymbol{S}=0 \tag{3.6}$$

will be satisfied provided Poynting's vector $\boldsymbol{S}$ is defined by

$$\boldsymbol{S}=\frac{1}{\mu_0}(\boldsymbol{E}_e\times\boldsymbol{B}_e+\boldsymbol{E}_m\times\boldsymbol{B}_m)\,. \tag{3.7}$$

Finally let us give the expression for the density of field momentum $\boldsymbol{p}$. Again we can use standard derivations ([5]) to show that $\boldsymbol{p}$ is given by

$$\boldsymbol{p}=\varepsilon_0\boldsymbol{E}_e\times\boldsymbol{B}_m\,. \tag{3.8}$$

Indeed one can see that $\boldsymbol{f}+\partial\boldsymbol{p}/\partial t$, where $\boldsymbol{f}$ is the density of force, *i.e.* the integrand in the r.h.s. of eq. (3.1), is equal to the divergence of a rather uninspiring dyadic, which we do not feel necessary to write down here. Since there is momentum in the field, Newton's third law is no longer true; for example, let us take an electric charge q moving at right angles towards a straight wire

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([4]) W. Panofsky and M. Phillips: *Classical Electricity and Magnetism*, Chap. 10 (Reading, Mass., 1962).

([5]) J. D. Jackson: *Classical Electrodynamics*, Chap. 6 (New York, 1962).

carrying a current I. This charge will experience a force $\boldsymbol{F} = q\boldsymbol{v} \times \boldsymbol{B}_m$, while the total force on the wire is zero (although there is of course a nonzero torque) (Fig. 1). This example shows that Newton's third law, if it is compatible with Galilean invariance, is not required by it.

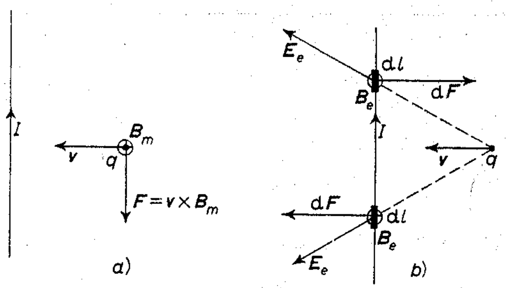

Fig. 1. – The reciprocal action of a charge in motion and a current-carrying straight wire: *a*) the wire exerts a neat force on the charge, *b*) the charge exerts a zero force (but nonzero torque) on the wire.

We conclude this Section with two remarks: first there is no relation between the density of field momentum $\boldsymbol{p}$ and Poynting's vector $\boldsymbol{S}$, in contrast to the usual case. Secondly, the density of field momentum and energy, and also the total field momentum and energy, are separately Galilean invariant. This may look rather strange, since one could expect that they transform according to one of the two nontrivial Galilean limits of the Lorentz transformation (cf. (4.11)). However our theory is not really a limit of the usual one, but rather a combination of two limits, and there is perhaps no reason to expect the correct transformation law.

4. – Galilean invariance and the constitutive equations of the vacuum.

Up to now we have taken the modern point of view (6) according to which the fundamental fields are $\boldsymbol{E}$ and $\boldsymbol{B}$. However, in older presentations of electromagnetism (7), it was customary to introduce, in addition to $\boldsymbol{E}$ and $\boldsymbol{B}$, two other fields, namely the electric and magnetic excitations $\boldsymbol{D}$ and $\boldsymbol{H}$. We wish to show that in this formulation, all the breaking of Galilean invariance can be confined to the constitutive equations

$$\boldsymbol{D} = \varepsilon \boldsymbol{E}, \qquad \boldsymbol{B} = \mu \boldsymbol{H} \tag{4.1}$$

(6) R. P. FEYNMAN: *The Feynman Lectures in Physics*, Vol. 2 (Reading, Mass., 1965); see also the first quoted book in ref. (2).
(7) See *e.g.* ref. (3), Part I.

even in vacuum ($\varepsilon = \varepsilon_0$, $\mu = \mu_0$). We emphasize that in this Section our point of view is completely different from the previous one: we do not want to build a Galilean invariant theory, but rather to exhibit how the requirement of Galilean invariance may lead to the hypothesis of an absolute reference frame. The fields $(\boldsymbol{E}, \boldsymbol{B}; \boldsymbol{H}, \boldsymbol{D})$ obey the usual equations (7)

$$\left\{\begin{aligned} \nabla\cdot\boldsymbol{D} &= \varrho\,, \\ \nabla\times\boldsymbol{E} &= -\frac{\partial\boldsymbol{B}}{\partial t}\,, \end{aligned}\right. \qquad \left\{\begin{aligned} \nabla\cdot\boldsymbol{B} &= 0\,, \\ \nabla\times\boldsymbol{H} &= \frac{\partial\boldsymbol{D}}{\partial t} + \boldsymbol{j}\,, \end{aligned}\right. \tag{4.2}$$

where

$$\frac{\partial\varrho}{\partial t} + \nabla\cdot\boldsymbol{j} = 0\,. \tag{4.3}$$

The Lorentz force is given by

$$\boldsymbol{F} = \int \mathrm{d}^3 r\,[\varrho(r)\boldsymbol{E}(r) + \boldsymbol{j}(r)\times\boldsymbol{B}(r)]\,. \tag{4.4}$$

It is remarkable that *these equations are Galilean invariant* with respect to the following transformation laws:

$$\left\{\begin{aligned} \boldsymbol{E}' &= \boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B}\,, \\ \boldsymbol{B}' &= \boldsymbol{B}\,, \end{aligned}\right. \qquad \left\{\begin{aligned} \boldsymbol{D}' &= \boldsymbol{D}\,, \\ \boldsymbol{H}' &= \boldsymbol{H} - \boldsymbol{v}\times\boldsymbol{D}\,, \end{aligned}\right. \tag{4.5}$$

$$\left\{\begin{aligned} \varrho' &= \varrho\,, \\ \boldsymbol{j}' &= \boldsymbol{j} - \boldsymbol{v}\varrho\,, \end{aligned}\right. \tag{4.6}$$

since these entail $\boldsymbol{F}' = \boldsymbol{F}$.

Notice that the continuity equation is satisfied, and that the current $\boldsymbol{j}$ may correspond to a transport of charge. Poynting's theorem holds in the conventional form

$$H\cdot\frac{\partial\boldsymbol{B}}{\partial t} + E\cdot\frac{\partial\boldsymbol{D}}{\partial t} + \boldsymbol{E}\cdot\boldsymbol{j} + \nabla\cdot(\boldsymbol{E}\times\boldsymbol{H}) = 0\,. \tag{4.8}$$

The energy density is given by

$$u = \tfrac{1}{2}\boldsymbol{E}\cdot\boldsymbol{D} + \tfrac{1}{2}\boldsymbol{B}\cdot\boldsymbol{H} \tag{4.9}$$

and the momentum density by

$$\boldsymbol{p} = \boldsymbol{D}\times\boldsymbol{B}\,. \tag{4.10}$$

It is quite interesting to observe that $(u, \boldsymbol{p})$ now obeys the Galilean transformation law for energy-momentum, written for massless particles (¹):

(4.11)
$$\begin{cases} u' = u - \boldsymbol{p}\cdot\boldsymbol{v}\,, \\ \boldsymbol{p}' = \boldsymbol{p} \end{cases}$$

(for massive bodies

(4.12)
$$\begin{cases} u' = u - \boldsymbol{p}\cdot\boldsymbol{v} + \tfrac{1}{2} d v^2\,, \\ \boldsymbol{p}' = \boldsymbol{p} - d\boldsymbol{v} \end{cases}$$

where d is the mass density (¹)). Since the volume is Galilean invariant, the transformation law for energy and momentum densities is the same as that of energy and momentum. One can say from (4.11) that the Galilean theory remembers the existence of photons!

Up to now, everything is quite nice, but the constitutive equations (4.1) are immediately seen to break Galilean invariance as they are inconsistent with the transformation laws ((4.5), (4.6)). The only way out (apart from relativity, of course!) is to assume that the constitutive equations hold only in some particular frame of reference, and this brings us at once to the ether theory. This Section shows that the following statement, « Maxwell's equations are incompatible with Galilean invariance », is a little bit loose. It is a perfectly valid statement provided one works with the fields $\boldsymbol{E}$ and $\boldsymbol{B}$ alone, but if one works with the excitation fields $\boldsymbol{D}$ and $\boldsymbol{H}$ also (which after all was the custom at the beginning of the century!) the structures (4.2) of Maxwell's equations and (4.4) of the Lorentz force are entirely compatible with Galilean relativity. It is then only at the level of the constitutive equations (4.1) that Galilean invariance breaks down.

5. – Conclusions.

Let us conclude by coming back to our previous view-point and discuss only Galilean invariant theories. What we have shown is that the following three assumptions are inconsistent:

1) Galilean invariance;

2) continuity equation: $\nabla\cdot\boldsymbol{j} = -\dfrac{\partial\varrho}{\partial t} \neq 0$;

3) magnetic forces between electric currents.

In the electric limit we keep 1) and 2), while in the magnetic limit we assume 1) and 3) to hold true. We have also shown that it is possible to combine both limits to build a Galilean invariant theory of electromagnetism. This theory looks complicated and awkward when compared to the usual one, although it is able to embody a large class of experimental facts.

The Galilean limits cannot be obtained by letting c go to infinity in the equations, without being very cautious [8].

The effect of the limiting process, as a matter of fact, depends upon the system of units. Any system, such as the GGS one, where c enters in the very definition of the units is bound to give inconsistent results. On the other hand, in an MKSA-type system, the initially « independent » coefficients ε_0 and μ_0 turn out to be related through the formula $\varepsilon_0\mu_0 c^2 = 1$. This is sufficient to show that Maxwell's equations cannot have a nonrelativistic limit ($c\to\infty$) where ε_0 and μ_0 both remain finite. The possibility of keeping one of them finite, at will, implies the existence of the two Galilean limits already recognized. In fact the following prescription yields the « magnetic » limit: express the relativistic theory in terms of the usual $\boldsymbol{E}$ and $\boldsymbol{B}$, keeping μ_0 but eliminating ε_0 (written instead $1/\mu_0 c^2$), then let c go to infinity. To obtain the « electric » limit, express the relativistic theory in terms of $\boldsymbol{E}$ and a redefined magnetic field $\widetilde{\boldsymbol{B}} = c^2\boldsymbol{B}$ (see also Appendix A), keeping ε_0 but eliminating μ_0 (written $1/\varepsilon_0 c^2$), then let $c\to\infty$:

It is straightforward to check that this prescription gives the correct limits for Maxwell's equations, the Lorentz force, the field transformation law, Poynting's vector and the field energy and momentum.

* * *

We are grateful to J. DONOHUE for his careful reading of the manuscript, and to J. P. PROVOST and J. L. MEUNIER for useful discussions.

APPENDIX A

Galilean electromagnetism and wave equations.

A direct construction of Galilean wave equations [9], based on the Bargmann-Wigner method, shows that free massless spin-one particles are characterized by a 6-component wave function, namely a pair of vector fields (L, M); they obey the following wave equation:

$$\text{(A.1)}\qquad \begin{cases} \nabla\cdot\boldsymbol{L} = 0\,, \\ \nabla\times\boldsymbol{L} = -\dfrac{\partial\boldsymbol{M}}{\partial t}\,, \end{cases} \qquad \begin{cases} \nabla\cdot\boldsymbol{M} = 0\,, \\ \nabla\times\boldsymbol{M} = 0\,, \end{cases}$$

(8) Relevant observations about this point have been made by H. BACRY and J. KUBAR-ANDRÉ: *Int. Journ. Theor. Phys.* (to be published), in a paper which is mainly devoted to magnetic monopoles in a Galilean framework. We intend to come back to this problem from the point of view developed in the present paper.

(9) J.-M. LÉVY-LEBLOND: *Comm. Math. Phys.*, **6**, 286 (1967). The present paper corrects and develops Sect. **6** of this article.

which is invariant under the Galilean transformation

$$\text{(A.2)} \qquad \begin{cases} \boldsymbol{L}' = \boldsymbol{L} - \boldsymbol{v} \times \boldsymbol{M}\,, \\ \boldsymbol{M}' = \boldsymbol{M}\,. \end{cases}$$

The unicity of this description might be thought to contradict the existence of two Galilean limits for the relativistic theory. However both these theories (for free fields) may be cast in the standard form (A.1). The identification

$$\boldsymbol{L} = \boldsymbol{E}_m\,, \qquad \boldsymbol{M} = \boldsymbol{B}_m\,,$$

immediately converts the general equations (A.1) into those of the magnetic limit (2.14). By setting $\boldsymbol{L} = \boldsymbol{B}_e/\varepsilon_0\mu_0$, $\boldsymbol{M} = -\boldsymbol{E}_e$ one instead recovers the electric limit (2.7). The interchange of the two limits by the substitution ($\boldsymbol{B}_m \leftrightarrow -\boldsymbol{E}_e$, $\boldsymbol{E}_m \leftrightarrow \boldsymbol{B}_e/\varepsilon_0\mu_0$) is obviously linked to the invariance of the original Maxwell's equations under the substitution ($\boldsymbol{B} \leftrightarrow -\boldsymbol{E}$, $\boldsymbol{E} \leftrightarrow \boldsymbol{B}/\varepsilon_0\mu_0$) .

APPENDIX B

We exhibit very briefly two Galilean invariant theories, in which one introduces two kinds of currents, $(\varrho_e, \boldsymbol{j}_e)$ and $(\varrho_m, \boldsymbol{j}_m)$, but only one kind of electromagnetic field $(\boldsymbol{E}, \boldsymbol{B})$.

a) Improved electric limit:

$$\text{(B.1)} \qquad \begin{cases} \nabla\cdot\boldsymbol{E} = \varrho_e/\varepsilon_0\,, \\ \nabla\times\boldsymbol{E} = 0\,, \end{cases} \qquad \begin{cases} \nabla\cdot\boldsymbol{B} = 0\,, \\ \nabla\times\boldsymbol{B} = \varepsilon_0\mu_0\dfrac{\partial\boldsymbol{E}}{\partial t} + \mu_0(\boldsymbol{j}_e + \boldsymbol{j}_m)\,, \end{cases}$$

$$\text{(B.2)} \qquad \boldsymbol{F} = \int \mathrm{d}^3 r\,(\varrho_e\boldsymbol{E} + \varrho_m\boldsymbol{E} + \boldsymbol{j}_m\times\boldsymbol{B})\,.$$

b) Improved magnetic limit:

$$\text{(B.3)} \qquad \begin{cases} \nabla\cdot\boldsymbol{E} = (\varrho_e + \varrho_m)/\varepsilon_0\,, \\ \nabla\times\boldsymbol{E} = -\dfrac{\partial\boldsymbol{B}}{\partial t}\,, \end{cases} \qquad \begin{cases} \nabla\cdot\boldsymbol{B} = 0\,, \\ \nabla\times\boldsymbol{B} = \mu_0\boldsymbol{j}_m\,, \end{cases}$$

$$\text{(B.4)} \qquad \boldsymbol{F} = \int \mathrm{d}^3 r\,(\varrho_e\boldsymbol{E} + \boldsymbol{j}_e\times\boldsymbol{B} + \boldsymbol{j}_m\times\boldsymbol{B})\,.$$

The interested reader will easily find which physical effects are taken into account by eqs. (B.1), (B.2) and (B.3), (B.4) respectively.

APPENDIX C

Relativity and spin-orbit coupling.

The spin-orbit coupling in atomic physics is conventionally derived through a « nonrelativistic » reasoning; one argues that the static electric field of the nucleus gives rise to a magnetic field in the reference frame of a moving electron, the magnetic moment of which then interacts with this field. Unfortunately the coupling thus calculated is half the real one and this is explained by the Thomas precession, usually presented as a « purely relativistic » effect. Now, there is some difficulty, to say the least, in understanding how a relativistic effect may induce a factor of 2, independently of the value of c! Moreover, the expression for this coupling energy contains a factor c^{-2} which seems rather strange for a nonrelativistic effect. Nevertheless, the conventional argument seems to be borne out by our consistently Galilean discussion of Subsect. 3'2 i) which yields indeed the usual, twice too small, « nonrelativistic » coupling. However, this is but an illusion; the theory of Sect. 3 is not a Galilean limit of the Maxwell theory. As a consequence, the $\varepsilon_0\mu_0$ factor in the spin-orbit coupling cannot be compared with the c^{-2} factor of the Thomas term deriving from relativistic kinematics. On the other hand, from the point of view of the Galilean limits of the Maxwell theory, there is no spin-orbit coupling at all. Indeed, we need to use here the « electric » limit to account for the electrostatic field of the nucleus. Now, in that limit (Sect. 2'2), the magnetic field induced in a moving reference frame (see eqs. (2.7)) does not act on currents or magnets (see eq. (2.12)) as we have stressed; the magnetic moment of the electron does not interact with it. In that sense, the induction of the magnetic field is as relativistic an effect as the Thomas precession, and the spin-orbit coupling should be viewed as a 100%, rather than 50%, relativistic property.

● RIASSUNTO (*)

Si indaga sulle teorie elettromagnetiche non relativistiche consistenti, mettendo in evidenza le esigenze della relatività galileiana. Si dimostra che le equazioni di Maxwell ammettono due possibili limiti non relativistici, che sono responsabili, rispettivamente, degli effetti elettrici e magnetici. Si costruisce poi una teoria galileiana che unisce queste due teorie, e che può includere un'ampia classe di fatti sperimentali. Si prova, come risultato, che parecchi effetti cosiddetti « relativistici » necessitano di una nuova valutazione, o, per lo meno, di una più attenta discussione. Infine, si dimostra precisamente come l'antica formulazione della teoria elettromagnetica, in termini di forze del campo e di eccitazioni del campo, si scontra con la relatività galileiana solamente nelle sue equazioni essenziali, portando all'idea di un mezzo di riferimento privilegiato (l'etere) o alla relatività di Einstein!

(*) *Traduzione a cura della Redazione.*