"VIOLATING" CLAUSER-HORNE INEQUALITIES WITHIN CLASSICAL MECHANICS

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Abstract. Some authors have raised the question whether the probabilities stemming from a quantum mechanical computation are entitled to enter the Bell and the Clauser-Horne inequalities. They have remarked that if the quantum probabilities are given the status of conditional ones and the statistics for the various settings of the detectors in a given experiment is properly kept into account, the inequalities happen to be no longer violated. In the present paper a classical simile modeled after the quantum mechanical instances is closely scrutinised. It is shown that the neglect of the conditional character of the probabilities in the classical model leads not only to "violate" the Clauser-Horne inequalities, but also to contradict the very axioms of classical probability theory.

1. Introduction

Many years have elapsed since Bell [1], [2] nailed down his prolegomena to any future hidden-variables theory, and Horne and Clauser worked out their inequalities [3], that shifted the whole issue from the realm of the *Gedankenexperimente* to the optical laboratories. And from these workshops of empirical evidence the response came, more and more convincing with the lapse of the years and the consequent refinement of the techniques for experimentation, aimed at closing all the possible loopholes [4], [5], [6]. Needless to say, the experiments without exceptions have confirmed, although not to everybody's satisfaction, the predictions stemming from quantum mechanics. Had not this been the case, theoretical physicists would have found something very interesting to do during the last years. But alas, quantum mechanics works finely even with these pairs of photons, whatever a photon may turn out to be [1]. Under this respect, the above mentioned experiments did not have much to add to what was already common wisdom for the overwhelming majority of physicists since many decades: quantum mechanics is the theory of choice for describing the microscopic reality in its interaction with macroscopic detectors.

But of course this is only one aspect of the question. The reason why the outcome of such delicate experiences is scrutinised by the theoreticians with so much attention depends also on their epistemological charme: not only is quantum mechanics once more confirmed by the experiments, but also (apart from residual loopholes) the so called local hidden-variables theories appear deprived of any residual hope for challenging quantum mechanics as the sole ruler of the microscopic realm [3].

Key words and phrases. Classical mechanics, quantum theory.

[1] In a letter of December 12, 1951 Albert Einstein wrote to his long-time friend Michele Besso: "The whole fifty years of conscious brooding have not brought me nearer to the answer to the question 'What are light quanta?'. Nowadays every scalawag believes that he knows what they are, but he deceives himself." [7] This English translation of the excerpt can be found in an article by J. Stachel [8].

At first sight, one does not grasp wherefrom the pressure for settling so abstract a question could come, for no credible pretenders to the role presently kept by quantum theory have emerged up to now. For this reason the attempt at outlining *a priori* the features of such would be pretenders is fraught with the unavoidable dangers of vagueness. Despite these risks, much effort has been spent for endowing the phantom pretenders with formal attributes that could allow for more and more general arguments against them. Much less work has been done (with some notable exceptions, like the ones accounted for in [9], [10]) for understanding what the good old classical physics [2] might have to suggest, through the analysis of particular instances, as hypothetical patterns of behaviour.

2. The Clauser-Horne inequalities

With the optical experiment in mind, Clauser and Horne have envisaged the behaviour that a wide class of stochastic local theories, designated by them as *objective local theories*, would display in the considered experimental circumstances. For completeness we summarise here the features of the model situation contemplated in [3]. A common source emits pairs of correlated entities that fly or propagate in opposite directions. When one of them reaches a detector, a characteristic feature of the entity, say the way it lies in a plane normal to the direction of motion, can be measured. Since the putative theory has to be an objective one, the entity is supposed to possess this feature in a way independent of any act of measurement. The overall behaviour of the single pair is characterized by a "hidden" parameter λ in the following sense: λ would specify the initial state of the pair and its subsequent development when the mutual interaction of the components has stopped, were it not for a residual element of uncertainty. The authors of [3] do not specify the nature of this uncertainty [3]. Anyway, a residual randomness [13] is left, maybe intrinsic, maybe just due to our laziness in investigating the system. Therefore it makes sense to speak of the conditional probability $p(A|\lambda)$ that the entity that propagates, say, to the left, when reaching the detector set on the left will display there a characteristic feature, for instance an angle A, when the hidden parameter has the value λ. Let $p(B|\lambda)$ be the conditional probability that the entity going to the right be detected to display as characteristic feature an angle B when the hidden parameter has the value λ. Clauser and Horne make an assumption, which they claim to be "a natural expression of a field-theoretical point of view, which in turn is an extrapolation from the common-sense view that there is no action at a distance". They stipulate that the local objective theories should obey the following factorisability condition:

$$p(A, B|\lambda) = p(A|\lambda)p(B|\lambda) \tag{2.1}$$

for the joint probability $p(A, B|\lambda)$ of detecting the left entity as displaying the angle A and the right entity as displaying the angle B when the hidden parameter has the value λ. Let A, A', and B, B' be the four angles that happen to be displayed by the propagating entities at their respective acts of detection when the hidden

[2] "*Die Physik der Modelle*", as Schrödinger nostalgically dubbed it [11].

[3] The relation of equivalence between deterministic and stochastic hidden-variables models was first elucidated [12] by A. Fine.

parameter has the value λ. It is an easy matter [3] to show that the inequalities

$$-1 \leq p(A|\lambda)p(B|\lambda) - p(A|\lambda)p(B'|\lambda) + p(A'|\lambda)p(B|\lambda) + p(A'|\lambda)p(B'|\lambda) - p(A'|\lambda) - p(B|\lambda) \leq 0 \tag{2.2}$$

must hold. By integrating $p(A|\lambda)$ and $p(A,B|\lambda)$ over λ with a suitably normalised weight function $\rho(\lambda)$ one defines the probability

$$p(A) \equiv \int \rho(\lambda)p(A|\lambda)d\lambda, \tag{2.3}$$

the joint probability

$$p(A,B) \equiv \int \rho(\lambda)p(A|\lambda)p(B|\lambda)d\lambda, \tag{2.4}$$

and eventually finds the now famous Clauser-Horne inequalities

$$-1 \leq p(A,B) - p(A,B') + p(A',B) + p(A',B') - p(A') - p(B) \leq 0 \tag{2.5}$$

as a necessary condition that all the objective local theories must fulfil. One cannot help agreeing with Clauser and Horne that the factorisability condition (2.1) is a necessary one in the objective local theories; one can add that, apart from the above mentioned exceptions [9], [10], *die Physik der Modelle* generally requires the satisfaction of such a condition, hence the general validity of the Clauser-Horne inequalities. When the quantum mechanical probabilities, let us say $q(A)$, $q(A,B)$, etc., calculated for particular instances dealing either with correlated spins or with correlated light quanta are substituted for the corresponding probabilities in (2.5), it is found that, if the angles are appropriately chosen, the resulting expression does not fulfil the inequalities. According to the established wisdom, this is the hallmark of quantum mechanics, well confirmed by the experimental facts (apart from residual loopholes), a unique feature that cannot be mimicked by classical physics, unless one contemplates either subtle enhancement processes connected *e.g.* with the existence of a zero-point electromagnetic field [9] or a nonlocal behaviour, like the one stemming from the Lorentz-Abraham-Dirac equation of motion for the classical electron [10].

In recent years, however, rather lonely but persistent voices have been heard [14], [15], [16], [17], [18], claiming *inter alia* that some not so venial sin is committed when substituting the quantum mechanical probabilities for the probabilities $p(A)$, $p(A,B)$ in (2.5), and that when the wrongdoing is amended, the Clauser-Horne inequalities happen to be no longer violated. More than seven decades have elapsed since the very notion of quantum probabilities has been hesitantly extracted [19], [20] by Born from Einstein's idea of the *Gespensterfeld*, and yet one feels some uneasiness when forced to confront a seemingly simple question like this. One would venture in this sort of discussion with a lighter heart, had the number of interpretations attached since then to quantum mechanics shrinked instead of increasing, as it has been the case, and provided that the quantum measurement problem had already been solved for good, *i.e. more relativistico*. While waiting for such occurrences, one still feels to walk on firmer ground if once more this admittedly old fashioned and just preliminary approach is attempted: seeking whether a more familiar simile, entirely grounded on classical physics and on classical probability theory, may provide some enlightenment.

3. A classical model that "violates" the inequalities

Let us consider an experimental device like the one drawn in Fig. 1, whose working is entirely accounted for by classical mechanics: two equal cylindrical bodies, "1" and "2", are thrown from the spatial origin of an inertial reference frame with opposite velocities $v_1 = -v_2$ and with opposite angular velocities $\omega_1 = -\omega_2$, all directed along the y axis, by the action of, say, a pressed and twisted spring interposed between them and suddenly released. The action of the spring lasts for a very short time, after which the two material bodies are free to run and to turn around their axes in the interior of a hollow cylinder of length l, kept at rest in the considered inertial reference system. The hollow cylinder is centered at the origin of the spatial coordinates and its axis lies in the y direction; its scope is to act as measuring device. To this end on its left half, along the whole span $-l/2 < y < 0$, two straight marks are engraved, whose projections on the x, z-plane happen to lay at the angles A and A' with respect to the x axis. Two straight marks are engraved also on the right half of the hollow cylinder, along its whole span $0 < y < l/2$, at the angles B and B' respectively. On the inner rims of the cylindrical bodies "1" and "2" two tiny marks m_1 and m_2 are impressed, and we shall abide to the rule that, after the pressing and twisting of the spring is accomplished, the two marks shall happen to coincide at some angle φ measured as previously in the x, z-plane. We shall also take care to load the spring always in the same way; the rotation angles of the two bodies, when they run from the center to the ends of the measuring device, shall be invariably $\gamma_1 = const. > 0$ for the body running to the left, and $\gamma_2 = -\gamma_1$ for the other one.

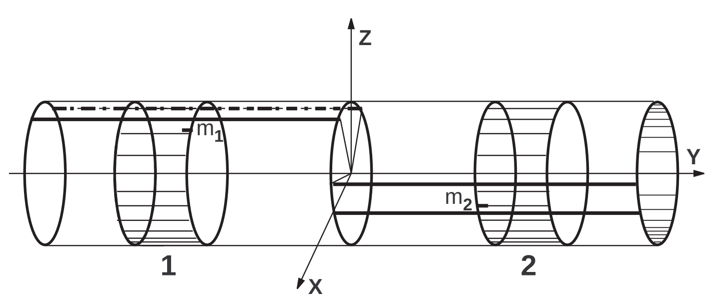

FIGURE 1. Schematic rendering of the experimental device aimed at "violating" the Clauser-Horne inequalities within classical mechanics. The heavy lines correspond to the marks engraved on the outer cylinder, respectively at the angles A, A' of its left half and at the angles B, B' of its right half. The spring, the mechanical constraint linking the bodies "1" and "2" and the pair of mechanical stops set at the angles, say, A and B are left to the imagination of the reader.

Let us indulge, with the given apparatus, in the following exercise of "experimental probability". We pick at random some angle φ from a uniform distribution that extends between 0 and 2π, we load and twist the spring interposed between the bodies as prescribed above, and we eventually set the coincident marks m_1 and

m_2 at the angle φ before releasing the spring. We then check whether the mark on body "1", that runs to the left, crosses or not the straight lines drawn at the angles A and A', and whether the mark on body "2" reaches or not the angles B and B'. After repeating *ab ovo* the whole procedure N times, the statistics of the experiment can be compiled, and the "experimental probability" can be eventually inferred. Of course nobody will really indulge in so trivial an experiment, since the physical situation is quite clear, and the classical probabilities $p(A)$, $p(A, B)$, etc. can be directly evaluated through a very simple geometric argument. Needless to say, when these probabilities are inserted in the Clauser-Horne inequalities (2.5), the latter happen to be fulfilled for all the possible choices of the angles A, A', and B, B'. Suppose however that we are caught by a virulent form of wishful thinking: we would like to see these inequalities violated, despite the fact that the instruments at our disposal are constituted by purely classical, objective and local entities. Therefore, in order to mimic the quantum mechanical behaviour, we modify the experimental device described above, and we conjure up an illusive "Verschränkung" by:

- connecting the bodies "1" and "2" through a mechanical constraint, that hinders their rotations when the mutual rotation angle reaches the value γ, in order to provide a not quite mysterious surrogate to the "spooky action at a distance",
- allowing for an active role of the measuring device, through mechanical stops placed in the interior of the hollow cylinder, respectively at an angle, say A, on the left side, acting in the span $-l/2 < y < 0$, and at an angle B on the right side, acting in the span $0 < y < l/2$. These stops are so contrived as to block the motion of the bodies "1" and "2" whenever the tiny marks m_1 and m_2 impressed on their inner rims respectively reach the above mentioned angles. It is intended that we can change the position of the mechanical stops from the pair of angles A, B to the pairs A, B', A', B and A', B', or even suppress one of the stops, while leaving the other one active and set just at one of the four positions mentioned above.

We test the modified device by choosing the angles as shown in Figure 2, *i.e.* we set $B' = 0$, $B = \vartheta$, $A' = \gamma$, $A = \gamma + \vartheta$, with $0 < \vartheta < \gamma$; the spring is loaded just in the way kept earlier, that entails a clockwise rotation of the body "2". It is our intention to perform again a sequel of trials with the start angle φ picked at random in a uniform distribution between 0 and 2π, as it was appropriate with the earlier version of the apparatus, but our plan meets with a certain difficulty: while previously just one series of trials, corresponding to just one experimental setup, was sufficient for inferring all the "experimental probabilities", the situation now is completely different. In order to gather the statistics required for inferring the joint probabilities four distinct sequences of trials are needed, one for each of the physically different setups that can occur when both mechanical stops are active. Furthermore, in order to infer the probabilities of the single occurrences, four additional sequences of trials seem needed, one for each of the physically different situations that can occur when one of the mechanical stops is removed. What we infer from the statistics of these eight independent sequences of trials, performed each one on a different physical system, are of course *conditional* probabilities. They are the probabilities for the bodies to reach certain angles when the stops are placed in a certain way. Also in this case there is no need to perform really the sequences

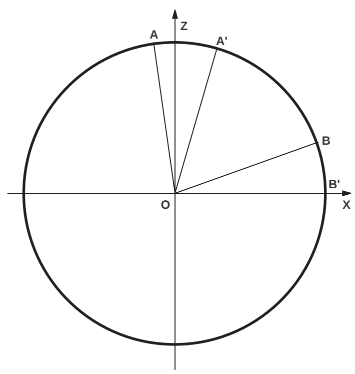

FIGURE 2. Representation on the trigonometric circle of the x, z-plane of the angles $B' = 0$, $B = \vartheta$, $A' = \gamma$, $A = \gamma + \vartheta$. They give the possible positions of the mechanical stops that allow for the "violation" of the Clauser-Horne inequalities described in the text.

of trials. Let $p(A, B|a, b)$ mean the conditional probability that the marks on the two bodies reach the angles A and B when the mechanical stops are both active and set just at the angles $a = A$, $b = B$, while $p(A|a)$ means the conditional probability that the mark on body "1" reach the angle A when only the mechanical stop on the left is active, and set at the angle $a = A$. Simple geometric arguments give

(3.1)
$$p(A, B|a, b) = p(A', B'|a', b') = \frac{\gamma}{2\pi},\ p(A, B'|a, b') = 0,\ p(A', B|a', b) = \frac{\gamma - \vartheta}{2\pi},$$

while

$$p(A|a) = p(A'|a') = p(B|b) = p(B'|b') = \frac{\gamma}{4\pi}. \tag{3.2}$$

It is quite clear that we should abstain from the inconsiderate act of plugging these quantities in the Clauser-Horne inequalities (2.5), since each one of them is a conditional probability referring to a different physical system, while the inequalities deal with the probabilites for just one physical system. If we insist in doing so, a nonsensical outcome must be expected, and is in fact obtained. The inequality (2.5) is "violated" on the right, despite the fact that classical mechanics fully accounts in an objective and local way for all the physical happenigs considered here. Moreover, through the same hocus-pocus also the inequality

$$-1 \leq p(A', B') - p(A', B) + p(A, B') + p(A, B) - p(A) - p(B') \leq 0, \tag{3.3}$$

that can be obtained from (2.5) by exchanging the angles with and without a prime, is "violated" on the right. By summing the inequalities stemming from the two "violations" one would get

$$2p(A, B) + 2p(A', B') - p(A) - p(A') - p(B) - p(B') > 0. \tag{3.4}$$

Of course one expects that in an entirely classical situation at least Bayes' axiom

$$p(A, B) = p(A)p(B|A) = p(B)p(A|B) \tag{3.5}$$

should be respected. But if we insist on this requirement and insert (3.5) in (3.4), we eventually reach an unquestionably absurd result: at least one of the four conditional probabilities $p(A|B)$, $p(B|A)$, $p(A'|B')$, $p(B'|A')$ should be > 1! By inserting the conditional probabilities (3.1) and (3.2) in the Clauser-Horne inequalities we have just committed a blunder. But the way out is not so arduous, and we could have spared the useless trials with one stop removed. The four trials with the two stops active are in fact sufficient for mastering the problem, if we confront it in the correct way. The measuring device, after the addition of the mechanical stops, is no longer a passive entity. Due to their presence the physical system under investigation is not constituted only by the bodies "1" and "2" and by the mechanical constraint that hinders mutual rotations of the latter larger than the angle γ. The physical system now includes the measuring device with its mechanical stops, and the probabilities that enter the Clauser-Horne inequalities shall deal with this larger system. Let us discard the trials that were performed with only one stop active, and consider the ratios between the number of trials performed with a certain setting of the two stops and the overall number of trials performed with the two stops both active. From these ratios we infer, in retrospect, the probabilities for the various settings of the stops. We call them $p(a, b)$, $p(a, b')$, $p(a', b)$, $p(a', b')$. Then the probabilities concerning the whole experiment, that we are entitled to insert in the Clauser-Horne inequalities, shall read:

$$\begin{aligned}
p(A, B) = p(A, B|a, b)p(a, b),\ & p(A, B') = p(A, B'|a, b')p(a, b'), \\
p(A', B) = p(A', B|a', b)p(a', b),\ & p(A', B') = p(A', B'|a', b')p(a', b'), \\
p(A) = p(A, B) + p(A, B'),\ & p(A') = p(A', B) + p(A', B'), \\
p(B) = p(A, B) + p(A', B),\ & p(B') = p(A, B') + p(A', B').
\end{aligned} \tag{3.6}$$

By inserting (3.6) in (2.5) the latter reduces to

$$-1 \leq -p(A, B'|a, b')p(a, b') - p(A', B|a', b)p(a', b) \leq 0 \tag{3.7}$$

which is of course not violated *in general* both on its right and on its left side.

4. Conclusion

We ask for the reader's forgiveness, because the scrutiny of the detailed working of our artful "Verschränkung" may well have looked a pedantic recitation of things so well known that their further recollection is just a waste of time. But what is given for granted in classical physics does not seem to be so well settled in quantum physics if, after so many years since the appearance [3] of the Clauser-Horne inequalities and after so many papers written on the subject, some authors [15], [16], [17] still felt urged to stand up and to remark that a major conceptual error, akin to the one expounded in the previous Section, has been committed in quantum physics. They have shown also that if the quantum probabilities are dealt with as *conditional* probabilities and the statistics of the use of the detectors is taken into account, quantum mechanics no longer happens to violate the inequalities in the situations that have been so carefully scrutinised, both by theoreticians and by experimentalists. The readings of the above mentioned papers and of the seminal work [14] by C. Brans have convinced us that even a recollection of the obvious, as it has been done with the classical simile of the present paper, could be helpful in intimating what might be the proper use of the Clauser-Horne inequalities in quantum mechanics.

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