HERMITIAN EXTENSION OF THE FOUR-DIMENSIONAL HOOKE'S LAW

S. ANTOCI

Abstract. It has been shown recently that the classical law of elasticity, expressed in terms of the displacement three-vector and of the symmetric deformation three-tensor, can be extended to the four dimensions of special and of general relativity with a physically meaningful outcome. In fact, the resulting stress-momentum-energy tensor can provide a unified account of both the elastic and the inertial properties of uncharged matter. The extension of the displacement vector to the four dimensions of spacetime allows a further possibility. If the real displacement four-vector ξ_i is complemented with an imaginary part φ_i, the resulting complex "displacement" four-vector allows for a complex, Hermitian generalisation of the four-dimensional Hooke's law.

Let the complex, Hermitian "stress-momentum-energy" tensor density $\mathbf{T}^{ik}$ built in this way be subjected to the "conservation condition" $\mathbf{T}^{ik}_{;k} = 0$. It turns out that, while the real part of the latter equation is able to account for the motion of electrically charged, elastic matter, the imaginary part of the same equation can describe the evolution of the electromagnetic field and of its sources. The Hermitian extension of Hooke's law is performed by availing of the postulate of "transposition invariance", introduced in 1945 by A. Einstein for finding the nonsymmetric generalisation of his theory of gravitation of 1915.

1. Introduction

It is well known that A. Einstein spent the last decade of his life in the search of a non-symmetric extension of his symmetric, Riemannian theory of 1915 [1],[2]. The very idea of such a generalisation dates back to 1925 [3], when Einstein first tried to unify the description of gravitation and of electromagnetism through an extension of the Riemannian geometry built from a nonsymmetric fundamental tensor and a nonsymmetric affine connection. But relaxing the symmetry of the geometrical objects that enter the equations of general relativity means an unwelcome arbitrariness in the choice of the generalisation. In Einstein's words [4]:

> The introduction of non-symmetric fields meets with the following difficulty. If Γ^l_{ik} is a displacement field, so is $\tilde{\Gamma}^l_{ik}$ ($= \Gamma^l_{ki}$). If g_{ik} is a tensor, so is $\tilde{g}_{ik}$ ($= g_{ki}$). This leads to a large

Key words and phrases. General relativity, electromagnetism, Hooke's law.
Nuovo Cimento B, in press.

> number of covariant formations among which it is not possible to make a selection on the principle of relativity alone.

The way out from such an arbitrariness was envisaged by Einstein in the requirement of "transposition invariance" as a generalisation of the principle of symmetry. A tensorial expression built with the fundamental form g_{ik} and with the affine connection Γ^l_{ik} is said to be transposition invariant with respect to the pair of indices, say, p and q, if it is transformed into itself when one simultaneously substitutes $\tilde{g}_{ik}$ for g_{ik}, $\tilde{\Gamma}^l_{ik}$ for Γ^l_{ik} and then interchanges the indices p and q. The requirement of transposition invariance played a key rôle in obtaining what Einstein's considered the logically most satisfactory solution of his problem, *i.e.* the field equations of the metric-affine theory [2]:

$$g_{ik,l} - g_{nk}\Gamma^n_{il} - g_{in}\Gamma^n_{lk} = 0, \tag{1.1}$$

$$\Gamma^s_{[is]} = 0, \tag{1.2}$$

$$R_{(ik)} = 0, \tag{1.3}$$

$$R_{[ik],l} + R_{[kl],i} + R_{[li],k} = 0; \tag{1.4}$$

whether a given expression be symmetric or antisymmetric with respect to a pair of indices, say p and q, is henceforth denoted by enclosing the mentioned indices within respectively round or square brackets; R_{ik} means the usual Ricci tensor

$$R_{ik} = \Gamma^a_{ik,a} - \Gamma^a_{ia,k} - \Gamma^a_{ib}\Gamma^b_{ak} + \Gamma^a_{ik}\Gamma^b_{ab} \tag{1.5}$$

built with the nonsymmetric connection Γ^l_{ik}.

Einstein's theory of the nonsymmetric field came in two versions, according to whether the fundamental tensor g_{ik} was chosen to be real nonsymmetric or complex Hermitian. Much effort was done by many authors in order to grasp the physical meaning of the proposed field equations; they looked for solutions either by exact or by approximate methods that mimicked the ones used in general relativity. All these efforts, however, either implicity or explicitly allowed for singularities at the right-hand side of one or another of the equations (1.1)-(1.4), while Einstein believed that, at variance with what occurs with his theory of 1915, the new theory had to be considered complete, and only everywhere regular solutions could disclose its physical meaning. Therefore, by agreeing with Einstein's conviction, one can assert that no conclusion has been drawn up to now about the validity of the nonsymmetric theory; this is one of the unsolved problem left as a challenge to the skillness of future mathematicians.

In the meantime one may well explore whether the very concept of invariance under transposition, that Einstein found so helpful in choosing his non Riemannian geometry, can be of heuristic value in some down to earth instance, dealing with some well known chapters of classical physics. Without attempting to solve difficult equations, one can content himself with a preliminary, modest task: seeking whether up to now unrelated physical entities, whose mathematical representation happens to require respectively symmetric and antisymmetric tensors, can be given an at least formally unified description in terms of either nonsymmetric or Hermitian tensors.

2. The four-dimensional Hooke's law as starting point

By availing of Cartesian coordinates and of the three-dimensional tensor formalism, Hooke's law [5] can be written as:

$$\Theta^{\lambda\mu} = \frac{1}{2} C^{\lambda\mu\rho\sigma} (\xi_{\rho,\sigma} + \xi_{\sigma,\rho}), \tag{2.1}$$

where $\Theta^{\lambda\mu}$ is the three-dimensional tensor that defines the stresses arising in elastic matter due to its displacement, given by the three-vector ξ^ρ, from a supposedly relaxed condition, and $C^{\lambda\mu\rho\sigma}$ is the constitutive tensor whose build depends on the material features and on the symmetry properties of the elastic medium. It has been shown [6] that this law admits of a natural generalisation to the four-dimensions of the Riemannian spacetime, whose metric tensor be g_{ik}. From a formal standpoint, this extension is straightforward: one introduces a real four-vector field ξ^i, that aims at representing some four-dimensional displacement, and builds the four-dimensional, symmetric deformation tensor

$$S_{ik} = \frac{1}{2} (\xi_{i;k} + \xi_{k;i}), \tag{2.2}$$

where the semicolon indicates covariant differentiation performed with the Christoffel symbols calculated from g_{ik}. A four-dimensional stiffness tensor density $\mathbf{C}^{iklm}$ is then introduced; it will be real and symmetric with respect to both the first and the second pair of indices, since it will be used for producing the real symmetric stress-momentum-energy tensor density

$$\mathbf{T}^{ik} = \mathbf{C}^{iklm} S_{lm}, \tag{2.3}$$

through the generalisation of equation (2.1) to the four dimensions of spacetime. This generalisation happens to make physical sense, since it allows encompassing both inertia and elasticity in a sort of four-dimensional elasticity [6]. Let us consider a coordinate system such that, at a given event, g_{ik} reduce to the form, say:

$$g_{ik} = \eta_{ik} \equiv diag(1, 1, 1, -1), \tag{2.4}$$

while the Christoffel symbols are vanishing, and the components of the four-velocity of matter are:

$$u^1 = u^2 = u^3 = 0, \; u^4 = 1. \tag{2.5}$$

We imagine that in such a coordinate system we are able to measure, at the chosen event, the three components of the (supposedly small) spatial displacement of matter from its relaxed condition, and that we adopt these three numbers as the values taken by ξ^ρ in that coordinate system, while the reading of some clock ticking the proper time and travelling with the medium will provide the value of the "temporal displacement" ξ^4 in the same coordinate system. By applying this procedure to all the events of the manifold where matter is present and by reducing the collected data to a common, arbitrary coordinate system, we can define the vector field $\xi^i(x^k)$. From such a field we shall require that, when matter is not subjected to ordinary strain and is looked at in a local rest frame belonging to the ones defined above, it will exhibit a deformation tensor S_{ik} such that its only nonzero component will be $S_{44} = \xi_{4,4} = -1$. This requirement is met if we define the four-velocity of matter through the equation

$$\xi^i_{;k} u^k = u^i. \tag{2.6}$$

The latter definition holds provided that

$$det(\xi^i_{;k} - \delta^i_k) = 0, \tag{2.7}$$

and this shall be one equation that the field ξ^i must satisfy; in this way the number of independent components of ξ^i will be reduced to three. A four-dimensional stiffness tensor C^{iklm} endowed with physical meaning can be built as follows. We assume that in a locally Minkowskian rest frame the only nonvanishing components of C^{iklm} are: $C^{\lambda\nu\sigma\tau}$, with the meaning of ordinary elastic moduli, and

$$C^{4444} = -\rho, \tag{2.8}$$

where ρ measures the rest density of matter. We need defining the four-dimensional stiffness tensor in an arbitrary co-ordinate system; this task is easily accomplished if the unstrained matter is isotropic when looked at in a locally Minkowskian rest frame. Let us define the auxiliary metric

$$\gamma^{ik} = g^{ik} + u^i u^k; \tag{2.9}$$

then the part of C^{iklm} accounting for the ordinary elasticity of the isotropic medium can be written as [7]

$$C^{iklm}_{\mathrm{el}} = -\lambda\gamma^{ik}\gamma^{lm} - \mu(\gamma^{il}\gamma^{km} + \gamma^{im}\gamma^{kl}), \tag{2.10}$$

where λ and μ are assumed to be constants. The part of C^{iklm} that accounts for the inertia of matter shall read instead

$$C^{iklm}_{\mathrm{in}} = -\rho u^i u^k u^l u^m. \tag{2.11}$$

The elastic part T^{ik}_{el} of the energy tensor is orthogonal to the four-velocity, as it should be [8]; thanks to equation (2.6) it reduces to

$$\begin{aligned} T^{ik}_{\mathrm{el}} = C^{iklm}_{\mathrm{el}} S_{lm} &= -\lambda(g^{ik} + u^i u^k)(\xi^m_{;m} - 1) \\ &-\mu[\xi^{i;k} + \xi^{k;i} + u_l(u^i\xi^{l;k} + u^k\xi^{l;i})], \end{aligned} \tag{2.12}$$

while, again thanks to equation (2.6), the inertial part of the energy tensor proves to be effectively so, since

$$T^{ik}_{\mathrm{in}} = C^{iklm}_{\mathrm{in}} S_{lm} = \rho u^i u^k. \tag{2.13}$$

The energy tensor defined by summing the contributions (2.12) and (2.13) encompasses both the inertial and the elastic energy tensor of an isotropic medium; when the macroscopic electromagnetic field is vanishing it should represent the overall energy tensor, whose covariant divergence must vanish according to Einstein's equations [9], [10]:

$$T^{ik}_{;k} = 0. \tag{2.14}$$

Imposing the latter condition allows one to write the equations of motion for isotropic matter subjected only to elastic strain [8]. We show this outcome in the limiting case when the metric is everywhere flat and the four-velocity of matter is such that u^ρ can be dealt with as a first order infinitesimal quantity, while u^4 differs from unity at most for a second order infinitesimal quantity. Also the spatial components ξ^ρ of the displacement vector and their derivatives are supposed to be infinitesimal at first order. An easy calculation [6] then shows that equation (2.7) is satisfied to the required first order, and that equations (2.14) reduce to the three equations of motion:

$$\rho\xi^\nu_{,4,4} = \lambda\xi^{\rho,\nu}_{,\rho} + \mu(\xi^{\nu,\rho} + \xi^{\rho,\nu})_{,\rho}, \tag{2.15}$$

and to the conservation equation

$$\{\rho u^4 u^k\}_{,k} = 0, \tag{2.16}$$

i.e., to the required order, they come to coincide with the well known equations of the classical theory of elasticity for an isotropic medium.

3. Hermitian extension of the four-dimensional Hooke's law

The extension of Hooke's law outlined in the previous Section has been a fruitful move, since it has allowed a unified account of inertia and of elasticity. But there is another aspect that deserves attention: having enlarged Hooke's law to the four dimensions of spacetime opens the way to this new question: is it useful, *i.e.*, does it make physical sense to look for either a nonsymmetric or a Hermitian extension of the four-dimensional Hooke's law?

Let us try the Hermitian version, and see how it can be formulated in a Riemannian spacetime whose metric, given a priori, is the real, symmetric tensor g_{ik}. We introduce a complex "displacement" four-vector

$$\omega^i \equiv \xi^i + i\varphi^i; \tag{3.1}$$

the real four-vectors ξ^i and φ^i enter respectively the real and the imaginary part of ω^i. By closely following the pattern of the real case, in lieu of (2.2) we introduce the Hermitian "deformation" tensor

$$S_{ik} = \frac{1}{2}(\omega^*_{i;k} + \omega_{k;i}); \tag{3.2}$$

the asterisk is henceforth used to denote complex conjugation. S_{ik} splits into a real, symmetric part $S_{(ik)}$, that can still be interpreted as a genuine deformation tensor:

$$S_{(ik)} = \frac{1}{2}(\xi_{i;k} + \xi_{k;i}) \tag{3.3}$$

and into an antisymmetric, purely imaginary contribution:

$$S_{[ik]} = \frac{i}{2}(\varphi_{k,i} - \varphi_{i,k}) \tag{3.4}$$

that immediately reminds us of Helmholtz' seminal attempt at producing a hydrodinamic *simile* of magnetism [11], and of the subsequent extension of the idea to the four dimensions of the theory of relativity. We assume that φ_i shall play the rôle of the electromagnetic four-potential and, starting from this apparently promising beginning, we try to figure out what the idea of a Hermitian generalisation of Hooke's law can entail from a physical standpoint.

Instead of the stiffness tensor density of equation (2.3), that is symmetric in both the first and the second pair of indices, one shall confront a complex "stiffness" tensor density $\mathbf{C}^{iklm}$, that will be Hermitian with respect to both the first and the second pair:

$$\mathbf{C}^{kilm} = \mathbf{C}^{*iklm} = \mathbf{C}^{ikml}. \tag{3.5}$$

This stiffness tensor density shall be contracted with the Hermitian deformation tensor S_{ik} to produce the Hermitian tensor density

$$\mathbf{T}^{ik} = \mathbf{C}^{iklm} S_{lm}. \tag{3.6}$$

One calls for now $\mathbf{T}^{ik}$ the Hermitian stress-momentum-energy tensor density, and asks that it fulfil the generalised "conservation" equation

$$\mathbf{T}^{ik}_{;k} = 0, \tag{3.7}$$

which of course entails severally:

$$\mathbf{T}^{(ik)}_{;k} = 0, \tag{3.8}$$

and

$$\mathbf{T}^{[ik]}_{,k} = 0. \tag{3.9}$$

A survey of the tensorial parts into which (3.6) can be split and expanded in keeping with the symmetry properties is now useful. The real, symmetric part of (3.6) can be written as:

$$\mathbf{T}^{(ik)} = \mathbf{C}^{(ik)(lm)} S_{(lm)} + \mathbf{C}^{(ik)[lm]} S_{[lm]}. \tag{3.10}$$

The interpretation of the first term at the right-hand side has been already provided, since $S_{(lm)}$ is just the four-dimensional, symmetric deformation tensor of Section 2; as recalled there, if $S_{(lm)}$ is contracted with the appropriate $\mathbf{C}^{(ik)(lm)}$, it can produce the material part of the symmetric energy tensor density, that accounts for both the inertia and the elasticity of matter. Let us call it $\mathbf{T}^{(ik)}_{\mathrm{m}}$.

Due to the presence of $S_{[lm]}$, the second term at the right-hand side of (3.10) awaits an electromagnetic interpretation. One tentatively sets

$$F_{ik} \equiv -2iS_{[ik]}, \tag{3.11}$$

where F_{ik} is the antisymmetric tensor used in electromagnetism, according to a long established convention, to encompass both the electric field $\vec{E}$ and the magnetic induction $\vec{B}$. A symmetric energy tensor density $\mathbf{T}^{(ik)}_{\mathrm{em}}$ for the electromagnetic field in matter, like *e.g.* Abraham's tensor density [12], [13] can obviously be recast in the form[1]

$$\mathbf{T}^{(ik)}_{\mathrm{em}} = \mathbf{C}^{(ik)[lm]} S_{[lm]}; \tag{3.12}$$

therefore the overall $\mathbf{T}^{(ik)}$ is apt to summarise the energy tensor density of elastic matter and of the electromagnetic field. Equating to zero the four-divergence of this density, as done in (3.8), would produce four equations for the motion of possibly charged elastic matter under the influence of an electromagnetic field.

We consider now the imaginary, antisymmetric part of $\mathbf{T}^{ik}$, that can be written as

$$\mathbf{T}^{[ik]} = \mathbf{C}^{[ik](lm)} S_{(lm)} + \mathbf{C}^{[ik][lm]} S_{[lm]}. \tag{3.13}$$

The second term at the right-hand side of this equation naturally fits the physical picture already emerged from the analysis of $\mathbf{T}^{(ik)}$. In fact, since $-2iS_{[ik]}$ has already been identified in (3.11) as representing the electric field $\vec{E}$ and the magnetic induction $\vec{B}$, one is led to pose

$$\mathbf{H}^{ik} \equiv -i\mathbf{T}^{[ik]}_{\mathrm{f}} \equiv -i\mathbf{C}^{[ik][lm]} S_{[lm]}, \tag{3.14}$$

i.e. to read off the second term at the right-hand side of (3.13) the so called constitutive equation of electromagnetism [14]. This equation defines the electric displacement $\vec{D}$ and of the magnetic field $\vec{H}$, summarised by the four-tensor density $\mathbf{H}^{ik}$, in terms of $\vec{E}$, $\vec{B}$, and of whatever fields may enter $\mathbf{C}^{[ik][lm]}$. For definiteness, let us remind an example of the constitutive equation, valid when matter is homogeneous and isotropic in its local rest frame [13]:

$$\mu H^{ik} = \left[g^{il} - (\epsilon\mu - 1)u^i u^l\right]\left[g^{km} - (\epsilon\mu - 1)u^k u^m\right] F_{lm}; \tag{3.15}$$

here ϵ is the dielectric constant and μ is the magnetic permeability. This constitutive equation has just the form (3.14).

If only the second term $\mathbf{C}^{[ik][lm]} S_{[lm]}$ where present at the right-hand side of (3.13), the imaginary part (3.9) of the "conservation" equation would entail $\mathbf{H}^{ik}_{\ ,k} = 0$, *i.e.* the electromagnetic field would be sourceless, and our description of matter would be defective. The Hermitian extension of the four-dimensional Hooke's law however provides a solution to this problem

[1]It may be objected that this way of writing the electromagnetic energy tensor is artificial, since the four-potential φ_i enters not only $S_{[lm]}$, but also the "stiffness" factor $\mathbf{C}^{(ik)[lm]}$. However the previous Section showed that a similar occurrence already happens with the displacement vector ξ_i in $\mathbf{T}^{(ik)}_{\mathrm{m}}$ as soon as one abandons the non-relativistic regime.

through the first term at the right-hand side of (3.13). Let us define:

$$\mathbf{P}^{ik} \equiv i\mathbf{T}^{[ik]}_{\mathrm{ch}} \equiv i\mathbf{C}^{[ik](lm)}S_{(lm)}, \;\; \mathbf{j}^i \equiv \mathbf{P}^{ik}_{\;\;,k}, \tag{3.16}$$

where $\mathbf{j}^i$ is a conserved quantity, since it is the divergence of an antisymmetric tensor density:

$$\mathbf{j}^i_{\;,i} = 0. \tag{3.17}$$

Equation (3.9) can be rewritten as

$$\mathbf{T}^{[ik]}_{\mathrm{ch}\;\;,k} + \mathbf{T}^{[ik]}_{\mathrm{f}\;\;,k} = 0, \tag{3.18}$$

and, due to the definitions (3.14) and (3.16), it entails:

$$\mathbf{H}^{ik}_{\;\;,k} = \mathbf{j}^i, \tag{3.19}$$

i.e. the validity of the inhomogeneous Maxwell's equation.

4. Charge, like matter, exists since time elapses

We have completed the attribution of a tentative physical meaning to the four tensorial terms into which the Hermitian $\mathbf{T}^{ik}$ can be split according to the symmetry properties. Three of them, namely the material contribution $\mathbf{T}^{(ik)}_{\mathrm{m}}$, the electromagnetic energy tensor $\mathbf{T}^{(ik)}_{\mathrm{em}}$ and the term that provides for the constitutive equation of electromagnetism, are entities known since a long time. Their mathematical form and their physical meaning have been carefully investigated by generations of scholars. The very existence of the fourth one, $\mathbf{T}^{[ik]}_{\mathrm{ch}}$, is predicted by the Hermitian extension of the four-dimensional Hooke's law. It is certainly welcome from a physical standpoint. Its very build, however, would be a surprise, had we not already met in Section 2 with an occurrence that constitutes its symmetric counterpart. We have seen there that the extension of Hooke's law to the four dimensions of spacetime allows one to account, besides the ordinary elasticity, also for the very existence of inertial matter. This happens thanks to the completion of the displacement vector with a fourth component which, as shown in Section 2, has the meaning of displacement in time. In the same way we must expect that the term $\mathbf{C}^{[ik](lm)}S_{(lm)}$ shall account, besides the phenomenon of ordinary piezoelectricity, also for the very existence of the electric charge and current in unstrained matter. Through the generalised deformation tensor $S_{(lm)}$ the relativistic and Hermitian extension of Hooke's law relates the presence of both matter and charge to the lapsing of time.

5. Perspectives

One still knows nothing about the explicit expression of $\mathbf{C}^{[ik](lm)}$, and it is fully premature to address here the enormous task of providing models that best suit the manifold properties displayed by electricity in macroscopic matter. Of course one shall proceed in this undertaking by adhering to the pattern already followed with the other three terms that compose $\mathbf{C}^{iklm}$, *i.e.* one shall try to build this fourth term by availing only of the two four-vectors ξ_i, φ_i, of the scalar density ρ and of the metric tensor g_{ik} This is possible, since (2.6) allows expressing the four-velocity u^i through ξ_i and g_{ik}. The postulate of Hermitian symmetry will restrict the choice of the possible forms. In the present formulation g_{ik} is an entity prescribed from the outside, and we know from the condition (2.7) that ξ_i has only three independent components. Pending a detailed scrutiny of the Cauchy problem, we can notice that the complex equation $\mathbf{T}^{ik}_{\ ;k} = 0$ imposes eight conditions on the eight independent variables of our problem. The model of electrified, elastic matter provided by the Hermitian extension of the four-dimensional Hooke's law has therefore a chance to stand up as a complete model, in which the values of all the quantities accounting for the physical behaviour of both matter and the electromagnetic field can be determined at least in principle by solving, with given initial conditions, the equations of motion stemming from the Hermitian "conservation equation".

References

1. Einstein A., Ann. Math., **46** (1945) 578.
2. Einstein A. and Kaufman B., Ann. Math., **62** (1955) 128.
3. Einstein A., S. B. Preuss. Akad. Wiss., **22** (1925) 414.
4. Einstein A., *The Meaning of Relativity*, p. 149 (Princeton University Press, Princeton), 1953.
5. Hooke R., *Lectures de Potentia restitutiva, or of Spring* (Martyn, London) 1678.
6. Antoci S. and Mihich L., Nuovo Cimento B, **114** (1999) 873.
7. Choquet-Bruhat Y. and Lamoureux-Brousse L., C. R. Acad. Sc. Paris A, **276** (1973) 1317.
8. Cattaneo C., C. R. Acad. Sc. Paris A, **272** (1971) 1421.
9. Hilbert D., Gött. Nachr., Math.-Phys. Klasse (1915) 395.
10. Klein F., Gött. Nachr., Math.-Phys. Klasse (1917) 469.
11. Whittaker E., *A history of the theories of aether and electricity* (Nelson, London) 1951, p. 245.
12. Abraham M., Rend. Circ. Matem. Palermo, **28** (1909) 1.
13. Gordon W., Ann. Phys. (Leipzig), **72** (1923) 421.
14. Post E.J., *Formal Structure of Electromagnetics* (North-Holland Publishing Company, Amsterdam) 1962.

Dipartimento di Fisica "A. Volta" and I. N. F. M., Via Bassi 6, Pavia, Italy
E-mail address: Antoci@fisav.unipv.it