

REVISITING WEYL'S CALCULATION OF THE GRAVITATIONAL PULL IN BACH'S TWO-BODY SOLUTION

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ABSTRACT. When the mass of one of the two bodies tends to zero, Weyl's definition of the gravitational force in an axially symmetric, static two-body solution can be given an invariant formulation in terms of a force four-vector. The norm of this force is calculated for Bach's two-body solution, that is known to be in one-to-one correspondence with Schwarzschild's original solution when one of the two masses l, l' is made to vanish. In the limit when, say, $l' \rightarrow 0$, the norm of the force divided by l' and calculated at the position of the vanishing mass is found to coincide with the norm of the acceleration of a test body kept at rest in Schwarzschild's field. Both norms happen thus to grow without limit when the test body (respectively the vanishing mass l') is kept at rest in a position closer and closer to Schwarzschild's two-surface.

1. INTRODUCTION

It is well known since a long time (see *e.g.* ([1])) that a test body kept at rest in Schwarzschild's gravitational field undergoes a four-acceleration whose norm tends to infinity if the position of the body is closer and closer to the Schwarzschild two-surface. However the existence of this singular behaviour of the gravitational pull, despite the fact that it can be given an invariant description, has not generally aroused very much concern. By far greater attention has been directed to the features of the world line followed by a test body in free motion, and already in 1950 it has been shown ([2]) that, once the singularity in the components of the metric is removed through a coordinate transformation that is aptly non regular at the Schwarzschild surface, a radial timelike geodesic may reach the Schwarzschild singularity and "cross it without a bump". This early finding by Synge has been the turning point for the program of analytic extension [3],[4]. Of course, geodesic motion has a fundamental role in general relativity. Nevertheless, it is quite reassuring that the very notion of the force exerted on a test body that is kept at rest, that has played so fundamental a role in the development of physical knowledge, does find a meaningful, *i.e.* invariant definition in general relativity. It is less reassuring that the norm of that force may be allowed to grow without limit at some surface in the interior of a manifold meant to be a realistic model of physical occurrences, without providing a justification for this allowance in physical terms.

The definition of the gravitational force felt by a test body of unit mass kept at rest in a static field is connected to the geometric definition of the four-acceleration by way of hypothesis. It would be interesting to calculate this force in an invariant way without availing of this hypothesis: Einstein's equations alone should suffice for the task. In the present paper the norm of the force exerted on a test body in Schwarzschild's field is obtained by starting, in the footsteps of Weyl [5],[6], from a particular two-body solution of Einstein's equations calculated in 1922 by R. Bach [6].

2. BACH'S SOLUTION FOR TWO "POINT MASSES"

While the spherically symmetric field of a "Massenpunkt" was determined by K. Schwarzschild [7] soon after the discovery of the field equations of general relativity [8], [9]¹ the class of axially symmetric, static solutions, that could provide some indication about the gravitational pull in a two-body system, was later found [11], [12] by Weyl and by Levi-Civita. Despite the nonlinear structure of the field equations, Weyl succeeded in reducing the problem to quadratures through the introduction of his "canonical cylindrical coordinates". Let $x^0 = t$ be the time coordinate, while $x^1 = z$, $x^2 = r$ are the coordinates in a meridian half-plane, and $x^3 = \varphi$ is the azimuth of such a half-plane; then the line element of a static, axially symmetric field *in vacuo* can be tentatively written as:

$$(2.1) \quad ds^2 = e^{2\psi} dt^2 - d\sigma^2, \quad e^{2\psi} d\sigma^2 = r^2 d\varphi^2 + e^{2\gamma} (dr^2 + dz^2);$$

the two functions ψ and γ depend only on z and r . Remarkably enough, in the "Bildraum" introduced by Weyl ψ fulfils the potential equation

$$(2.2) \quad \Delta\psi = \frac{1}{r} \left\{ \frac{\partial(r\psi_z)}{\partial z} + \frac{\partial(r\psi_r)}{\partial r} \right\} = 0$$

(ψ_z , ψ_r are the derivatives with respect to z and to r respectively), while γ is obtained by solving the system

$$(2.3) \quad \gamma_z = 2r\psi_z\psi_r, \quad \gamma_r = r(\psi_r^2 - \psi_z^2);$$

due to the potential equation (2.2)

$$(2.4) \quad d\gamma = 2r\psi_z\psi_r dz + r(\psi_r^2 - \psi_z^2) dr$$

happens to be an exact differential.

Schwarzschild's original "Massenpunkt" solution [7] is recovered exactly when ψ is that solution of (2.2) corresponding to the Newtonian potential that one obtains if one segment of the z -axis is covered by matter with constant mass density [11]. Let $2l$ be the coordinate length of this segment, and let r_1 , r_2 be the "distances", calculated in the Euclidean way, of a point P with canonical coordinates z , r from the end points P_1 and P_2 of the

¹Schwarzschild actually worked with the next-to-last version of the theory [10], whose covariance was limited to unimodular transformations. As we shall see later, this fortuitous circumstance had momentous consequences.

segment, that lie on the symmetry axis at $z = z_1$ and at $z = z_2 = z_1 - 2l$ respectively. One finds

$$(2.5) \quad \psi = \frac{1}{2} \ln \frac{r_1 + r_2 - 2l}{r_1 + r_2 + 2l}; \quad \gamma = \frac{1}{2} \ln \frac{(r_1 + r_2)^2 - 4l^2}{4r_1 r_2}.$$

Through a coordinate transformation in the meridian half-plane one proves the agreement with Schwarzschild's result, if m is substituted for l . We draw the attention of the reader on the fact that the agreement occurs with Schwarzschild's original solution [7], not with what is called "Schwarzschild solution" in all the textbooks, but is in fact the spherically symmetric solution found by Hilbert through his peculiar choice of the radial coordinate [13]. In fact by setting

$$(2.6) \quad z - z_2 = l(1 - \cos \vartheta),$$

one finds that the spatial part $d\sigma^2$ of the square of the line element on the segment $P_2 P_1$ becomes

$$(2.7) \quad d\sigma^2 = 4l^2(d\vartheta^2 + \sin^2 \vartheta d\varphi^2),$$

i.e. it coincides with the square of the line element of the spherical two-surface that is present at $r = 0$ in Schwarzschild's true and original solution [7] of Einstein's equations. The solution given by (2.5) is in one-to-one correspondence with that solution, and it does not contain the "interior region" that Hilbert could not help finding [13] due to the particular way he kept in fixing the radial coordinate.

A static two-body solution is instead obtained if one assumes, like Bach did [6], that the "Newtonian potential" ψ is generated by matter that is present with constant mass density on two segments of the symmetry axis, like the segments $P_4 P_3$ and $P_2 P_1$ of Figure 1. We know already that the particular choice

$$(2.8) \quad \psi = \frac{1}{2} \ln \frac{r_1 + r_2 - 2l}{r_1 + r_2 + 2l} + \frac{1}{2} \ln \frac{r_3 + r_4 - 2l'}{r_3 + r_4 + 2l'},$$

will produce a vacuum solution to Einstein's field equations that reduces to Schwarzschild's original solution if one sets either $l = 0$ or $l' = 0$. Of course, due to the nonlinearity of (2.3) one cannot expect that γ will contain only the sum of the contributions

$$(2.9) \quad \gamma_{11} = \frac{1}{2} \ln \frac{(r_1 + r_2)^2 - 4l^2}{4r_1 r_2}, \quad \gamma_{22} = \frac{1}{2} \ln \frac{(r_3 + r_4)^2 - 4l'^2}{4r_3 r_4},$$

corresponding to the individual terms of the potential (2.8); a further term is present, that Bach called γ_{12} , and reads

$$(2.10) \quad \gamma_{12} = \ln \frac{lr_4 - (l' + d)r_1 - (l + l' + d)r_2}{lr_3 - dr_1 - (l + d)r_2} + c,$$

where c is a constant. Since γ must vanish at the spatial infinity, it must be $c = \ln[d/(l + l')]$. With this choice of the constant one eventually finds [6]

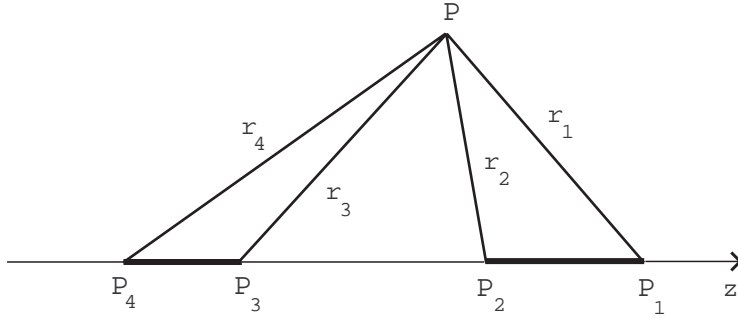


FIGURE 1. Representation in the canonical z, r half-plane of the mass sources for Bach's two-body solution. r_4, r_3 and r_2, r_1 are the "distances", calculated in the Euclidean way, of a point P from the end points of the two segments endowed with mass. $\overline{P_4P_3} = 2l', \overline{P_3P_2} = 2d, \overline{P_2P_1} = 2l$, again in coordinate lengths.

that the line element of the two-body solution is defined by the functions

$$\begin{aligned}
 e^{2\psi} &= \frac{r_1 + r_2 - 2l}{r_1 + r_2 + 2l} \cdot \frac{r_3 + r_4 - 2l'}{r_3 + r_4 + 2l'}, \\
 e^{2\gamma} &= \frac{(r_1 + r_2)^2 - 4l^2}{4r_1r_2} \cdot \frac{(r_3 + r_4)^2 - 4l'^2}{4r_3r_4} \\
 (2.11) \quad &\cdot \left(\frac{d(l' + d)r_1 + d(l + l' + d)r_2 - ldr_4}{d(l' + d)r_1 + (l + d)(l' + d)r_2 - l(l' + d)r_3} \right)^2.
 \end{aligned}$$

With these definitions for ψ and γ the line element (2.1) behaves properly at the spatial infinity and is regular everywhere, except for the two segments P_4P_3, P_2P_1 of the symmetry axis, where the sources of ψ are located, and also for the segment P_3P_2 , because there γ does not vanish as required, but takes the constant value

$$(2.12) \quad \Gamma = \ln \frac{d(l + l' + d)}{(l + d)(l' + d)},$$

thus giving rise to the well known conical singularity.

3. WEYL'S ANALYSIS OF THE STATIC TWO-BODY SOLUTIONS

Due to this lack of elementary flatness occurring on the segment P_3P_2 the solution is not a true two-body solution; nevertheless Weyl showed [6] that a regular solution could be obtained from it, provided that nonvanishing energy tensor density \mathbf{T}_i^k be allowed for in the space between the two bodies. In this way an axial force K is introduced, with the evident function of

keeping the two bodies at rest ² despite their mutual gravitational attraction. By providing a measure for K , Weyl provided a measure of the gravitational pull. Let us recall here Weyl's analysis [5],[6] of the axially symmetric, static two-body problem.

In writing Einstein's field equations, we adopt henceforth Weyl's convention for the energy tensor:

$$(3.1) \quad R_{ik} - \frac{1}{2}g_{ik}R = -T_{ik}.$$

Einstein's equations teach that, when the line element has the expression (2.1), \mathbf{T}_i^k shall have the form

$$(3.2) \quad \begin{pmatrix} \mathbf{T}_0^0 & 0 & 0 & 0 \\ 0 & \mathbf{T}_1^1 & \mathbf{T}_2^1 & 0 \\ 0 & \mathbf{T}_1^2 & \mathbf{T}_2^2 & 0 \\ 0 & 0 & 0 & \mathbf{T}_3^3 \end{pmatrix}$$

where

$$(3.3) \quad \mathbf{T}_1^1 + \mathbf{T}_2^2 = 0.$$

By introducing the notation

$$(3.4) \quad \mathbf{T}_3^3 = r\varrho', \quad \mathbf{T}_0^0 = r(\varrho + \varrho'),$$

Einstein's equations can be written as:

$$(3.5) \quad \Delta\psi = \frac{1}{2}\varrho, \quad \frac{\partial^2\gamma}{\partial z^2} + \frac{\partial^2\gamma}{\partial r^2} + \left\{ \left(\frac{\partial\psi}{\partial z} \right)^2 + \left(\frac{\partial\psi}{\partial r} \right)^2 \right\} = -\varrho';$$

$$(3.6) \quad \mathbf{T}_1^1 = -\mathbf{T}_2^2 = \gamma_r - r(\psi_r^2 - \psi_z^2), \quad -\mathbf{T}_1^2 = -\mathbf{T}_2^1 = \gamma_z - 2r\psi_r\psi_z.$$

Weyl shows that ϱ must be interpreted as mass density in the canonical space. To this end he considers the mass density distribution sketched in Figure 2, where ϱ is assumed to be nonvanishing only in the shaded regions labeled 1 and 2. According to (3.5) the potential ψ corresponding to this mass distribution can be uniquely split in two terms ψ_1 and ψ_2 , such that ψ_1 is a potential function that vanishes at infinity and is everywhere regular outside the region 1, while ψ_2 behaves in the same way outside the region 2. The asymptotic forms of ψ_1 and ψ_2 are such that

$$(3.7) \quad e^{2\psi_1} = 1 - \frac{m_1}{R} + \dots, \quad e^{2\psi_2} = 1 - \frac{m_2}{R} + \dots$$

where the mass coefficients m_1 and m_2 are given by the integral $\int \varrho dV = 2\pi \int \varrho r dr dz$, performed in the canonical space and extended to the appropriate shaded region. Outside the shaded regions one has $\varrho = 0$, but there

²If a metric is static the definition of rest with respect to that metric can be given in invariant form through the Killing vectors.

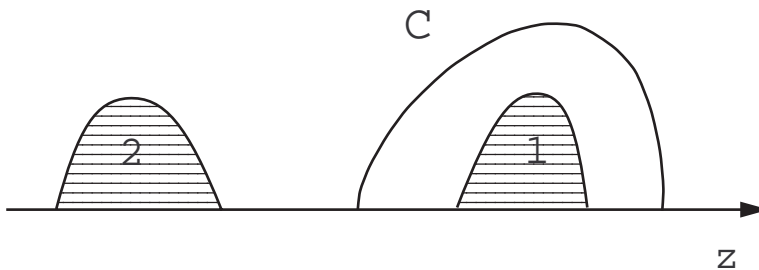


FIGURE 2. Representation in the canonical z, r half-plane of extended mass sources of a two-body solution.

shall be some region between the bodies, let us call it L' , where $\varrho = 0$ but $\mathbf{T}_i^k \neq 0$, since in a static solution of general relativity the gravitational pull shall be counteracted in some way. Weyl's procedure for determining \mathbf{T}_i^k in L' is the following. Suppose that $\mathbf{T}_i^k = 0$ outside a simply connected region L that includes both material bodies. Since ψ is known there, we can avail of (2.4), together with the injunction that γ vanish at infinity, to determine γ uniquely outside L . Within L' we can choose γ arbitrarily, provided that we ensure the regular connection with the vacuum region and the regular behaviour on the axis, *i.e.* γ vanishing there like r^2 . Since ψ is known in L' and γ has been chosen as just shown, we can use equations (3.5) and (3.6) to determine \mathbf{T}_i^k there.

If the material bodies include each one a segment of the axis, just as it occurs in Fig. 2, the force K directed along the z axis, with which the stresses in L' contrast the gravitational pull can be written as

$$(3.8) \quad K = 2\pi \int_C (\mathbf{T}_1^2 dz - \mathbf{T}_1^1 dr);$$

the integration path is along a curve C , like the one drawn in Fig. 2, that separates the two bodies in the meridian half-plane; the value of the integral does not depend on the precise position of C because, as one gathers from the definitions (3.5), (3.6):

$$(3.9) \quad \mathbf{T}_{1,1}^1 + \mathbf{T}_{1,2}^2 = 0$$

in the region L' . Since the region of the meridian half-plane where $\varrho = 0$ is simply connected, by starting from ψ and from the vacuum equation (2.4), now rewritten as:

$$(3.10) \quad d\gamma^* = 2r\psi_z\psi_r dz + r(\psi_r^2 - \psi_z^2)dr$$

one can uniquely define there the function γ^* that vanishes at the spatial infinity. In all the parts of the z axis where $\varrho = 0$ it must be $\gamma_z^* = 0$, $\gamma_r^* = 0$, hence $\gamma^* = \text{const.}$, $\gamma_r^* = 0$. In particular, in the parts of the axis that go to infinity one shall have $\gamma^* = 0$; let us call Γ^* the constant value assumed

instead by γ^* on the segment of the axis lying between the two bodies. The definitions (3.6) can now be rewritten as:

$$(3.11) \quad \mathbf{T}_1^1 = -\mathbf{T}_2^2 = \gamma_r - \gamma_r^*, \quad -\mathbf{T}_1^2 = -\mathbf{T}_2^1 = \gamma_z - \gamma_z^*,$$

and the integral of (3.8) becomes

$$(3.12) \quad \int_C (\mathbf{T}_1^2 dz - \mathbf{T}_1^1 dr) = \int_C (\gamma_z^* - \gamma_z) dz + (\gamma_r^* - \gamma_r) dr = \int_C d(\gamma^* - \gamma).$$

Since γ vanishes on the parts of the z axis where $\varrho = 0$, the force K that holds the bodies at rest despite the gravitational pull shall be

$$(3.13) \quad K = -2\pi\Gamma^*$$

with Weyl's definition (3.1) of the energy tensor. When the mass density ϱ has in the canonical space the particular distribution considered by Bach and drawn in Fig. 1, Γ^* is equal to Γ as defined by (2.12). The measure of the gravitational pull with which the two "material bodies" of this particular solution attract each other therefore turns out to be

$$(3.14) \quad K = 2\pi \ln \frac{(d+l)(d+l')}{d(d+l+l')}$$

in Weyl's units. This expression agrees with the Newtonian value when l and l' are small when compared to d , as expected.

4. FROM WEYL'S K TO A "QUASI" FORCE FOUR-VECTOR k_i

Despite its mathematical beauty, Weyl's definition of the gravitational pull for an axially symmetric, static two-body solution appears associated without remedy to the adoption of the canonical coordinate system. It is however possible to obtain through Weyl's definition of K , given by (3.8), a "quasi" four-vector k_i . In fact that expression can be rewritten as

$$(4.1) \quad K = \int_{\Sigma} \mathbf{T}_1^l df_{0l}^* \equiv \frac{1}{2} \int_{\Sigma} T_1^l \epsilon_{0lmn} df^{mn},$$

where ϵ_{klmn} is Levi-Civita's totally antisymmetric tensor and df^{mn} is the element of the two-surface Σ generated by the curve C through rotation around the symmetry axis. Since the metric is static it is possible to define invariantly a timelike Killing vector $\xi_{(t)}^k$ that correspond, in Weyl's canonical coordinates, to a unit coordinate time translation. Therefore (4.1) can be rewritten as

$$(4.2) \quad K = \frac{1}{2} \int_{\Sigma} \xi_{(t)}^k T_1^l \epsilon_{klmn} df^{mn}$$

by still using the canonical coordinates. Now the integrand is written as the "1" component of the infinitesimal covariant four-vector

$$(4.3) \quad \xi_{(t)}^k T_i^l \epsilon_{klmn} df^{mn},$$

but of course in general the expression

$$(4.4) \quad k_i = \frac{1}{2} \int_{\Sigma} \xi_{(t)}^k T_i^l \epsilon_{klmn} df^{mn}$$

will not be a four-vector, because the integration over Σ spoils the covariance. When evaluated in canonical coordinates, the nonvanishing components of k_i are $k_1 = K$ and

$$(4.5) \quad k_2 = 2\pi \int_C (\mathbf{T}_2^2 dz - \mathbf{T}_1^2 dr) = 2\pi \int_C (\gamma_r^* - \gamma_r) dz - (\gamma_z^* - \gamma_z) dr,$$

that however must vanish too, if k_i has to become a four-vector defined on the symmetry axis. But, as one sees from Weyl's analysis, we are at freedom to choose \mathbf{T}_i^k in L' as nonvanishing only in a tube with a very small³ yet finite coordinate radius that encloses in its interior the segment of the symmetry axis lying between the bodies; moreover, we can freely set $\gamma_z = \gamma_z^*$ within the tube. Under these conditions the second term of the integral (4.5) just vanishes, while the first one shall be very small, since the regularity of the surface Σ requires that the curve C approach the symmetry axis at a straight angle in canonical coordinates. By properly choosing \mathbf{T}_i^k we thus succeed in providing through equation (4.4) a quasi four-vector k_i whose components, written in Weyl's canonical coordinates, reduce in approximation to $(K, 0, 0, 0)$.

5. THE NORM OF THE FORCE IN BACH'S SOLUTION WHEN $2l' \rightarrow 0$

Having defined, with the above caveats, the quasi four-vector k_i along the segment of the symmetry axis between the two bodies, we can use its "quasi" norm to provide a measure of the force that opposes the gravitational pull. In the case of Bach's two-body solution, whose line element is defined in canonical coordinates by (2.1) and (2.11), that quasi norm reads

$$(5.1) \quad k \equiv (-k^i k_i)^{1/2} = 2\pi \ln \frac{(d+l)(d+l')}{d(d+l+l')} \cdot \left[\frac{r_1 - 2l}{r_1} \cdot \frac{r_4 - 2l'}{r_4} \right]^{1/2}$$

when measured in Weyl's units at a point of the symmetry axis for which $z_3 < z < z_2$. At variance with the behaviour of K , the quasi norm k depends on z , due to the term of (5.1) enclosed within the square brackets, that comes from $e^{2\psi}$. Let us evaluate this quasi norm divided by l' when $l' \rightarrow 0$, namely, the coefficient of the linear term in the McLaurin series expansion of k with respect to l' . Since Γ^* , now defined by the right hand side of (2.12), tends to zero when $l' \rightarrow 0$, while performing this limit one can also send to zero the radius of the very narrow tube considered in the previous section. Therefore

³This kind of procedure has been used to derive the equations of motion even for structured particles by Einstein, Infeld and Hoffman [14] and by Fock and Papapetrou (see [15]).

k_i can become a true four-vector and k can become a true norm in the above mentioned limit. With this proviso one finds the invariant result

$$(5.2) \quad \lim_{l' \rightarrow 0} \left[\frac{k}{l'} \right] = \left[\frac{\partial k}{\partial l'} \right]_{l'=0} = \frac{2\pi l}{d(d+l)} \left(\frac{r_1 - 2l}{r_1} \right)^{1/2}.$$

When $l' \rightarrow 0$ the line element of Bach's solution with two bodies tends to the line element defined by (2.1) and (2.5), that is in one-to-one correspondence with the line element of Schwarzschild's original solution [7]. Therefore the scalar quantity $[\partial k / \partial l']_{l'=0}$ evaluated at P_3 shall be the norm of the force per unit mass exerted by Schwarzschild's gravitational field on a test particle kept at rest at P_3 . Its value is obtained by substituting $2d + 2l$ for r_1 in (5.2). One finds

$$(5.3) \quad \left(\lim_{l' \rightarrow 0} \left[\frac{k}{l'} \right] \right)_{z=z_3} = \frac{8\pi l}{(2d + 2l)^{3/2}(2d)^{1/2}}.$$

If one solves Schwarzschild's problem in spherical polar coordinates r, ϑ, φ, t with three unknown functions of r , *i.e.* without fixing the radial coordinate, like Combridge and Janne did long ago [16],[17], one ends up to write de Sitter's line element [18]

$$(5.4) \quad ds^2 = -\exp \lambda dr^2 - \exp \mu [r^2(d\vartheta^2 + \sin^2 \vartheta d\varphi^2)] + \exp \nu dt^2$$

in terms of one unknown function $f(r)$. In fact λ, μ, ν are defined through this arbitrary function $f(r)$ and through its derivative $f'(r)$ as follows:

$$(5.5) \quad \exp \lambda = \frac{f'^2}{1 - 2m/f},$$

$$(5.6) \quad \exp \mu = \frac{f^2}{r^2},$$

$$(5.7) \quad \exp \nu = 1 - 2m/f.$$

Here m is the mass constant; of course the arbitrary function f must have the appropriate behaviour as $r \rightarrow \infty$. Schwarzschild's original solution [7] is eventually recovered [19],[20] by requiring that f be a monotonic function of r and that $f(0) = 2m$. Let us imagine that a test body be kept at rest in this field. With our symmetry-adapted coordinates, its world line shall be invariantly specified by requiring that the spatial coordinates r, ϑ, φ of the test body be constant in time. If

$$(5.8) \quad \alpha = (-a_i a^i)^{1/2}$$

is the norm of the acceleration four-vector

$$(5.9) \quad a^i \equiv \frac{du^i}{ds} + \Gamma_{kl}^i u^k u^l$$

along the world line of the test body, one finds:

$$(5.10) \quad \alpha = \frac{m}{f^{3/2}(f - 2m)^{1/2}}.$$

This norm is assumed by way of hypothesis to be equal to the norm of the force per unit mass needed for constraining the test particle to follow a world line of rest despite the gravitational pull of the Schwarzschild field [1]. The consistency of the hypothesis with Einstein's theory requires that α be equal to the scalar quantity $[\partial k / \partial l']_{l'=0, z=z_3}$ that provides the norm of the force per unit mass for Bach's solution in the test particle limit $l' \rightarrow 0$. This is indeed the case, since the functional dependence of (5.3) on the mass parameter l and on the coordinate distance $2d + 2l$ is the same as the functional dependence of (5.10) on the mass parameter m and on the function $f(r)$ with $f(0) = 2m$ introduced above. The extra constant 8π appearing in (5.3) is just due to Weyl's adoption of the definition (3.1) of the energy tensor.

For Schwarzschild's field, the definition of the norm of the force exerted on a test particle at rest obtained through the acceleration four-vector and the independent definition through the force that, in Bach's two-body solution, \mathbf{T}_i^k must exert to keep the masses at rest when $l' \rightarrow 0$ lead to one and the same result. In particular, both definitions show that the norm of the force per unit mass grows without limit as the test particle is kept at rest in a position closer and closer to Schwarzschild's two-surface.

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