A METHOD FOR FINDING SOLUTIONS OF THE HERMITIAN THEORY OF RELATIVITY WHICH DEPEND ON THREE CO-ORDINATES

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Abstract. A method is presented, which can generate solutions of the Hermitian theory of relativity from known solutions of the general theory of relativity, when the latter depend on three co-ordinates and are invariant under reversal of the fourth one.

Eine Methode zum Auffinden von Lösungen der Hermiteschen Relativitätstheorie, die von drei Koordinaten abhängen

Inhaltsübersicht. Es wird eine Methode vorgestellt, nach der Lösungen der Hermiteschen Relativitätstheorie aus bekannten Lösungen der Allgemeinen Relativitätstheorie gewonnen werden können, wenn diese von drei Koordinaten abhängen und bei Umkehrung der vierten invariant bleiben.

1. Introduction

In recent times several exact solutions for the field equations of the Hermitian theory or relativity [1] have been found. Some of them [2] have confirmed the result, reached in 1957 by approximation methods [3], that the theory entails the existence of confined charges, interacting mutually with forces which do not depend on the distance [4]. Other solutions, in which the antisymmetric part of the fundamental tensor g_{ik} obeys Maxwell's equations, just predict the equilibrium positions which are appropriate to electric charges and currents, so that it is reasonable to believe that Einstein's Hermitian theory of relativity can provide a unified description of gravodynamics, chromodynamics and electrodynamics [5].

It was observed that the solutions mentioned above can be constructed from particular solutions of the field equations of general relativity by following a certain procedure; the present paper shows that such an occurrence is not a pure accident, since a method exists, which can generate solutions of the Hermitian theory of relativity from given solutions of general relativity, when the latter depend on three co-ordinates and are invariant under reversal of the fourth one.

Annalen der Physik, **44** (1987) 297.

2. A Method for Finding Solutions

Let us consider a Hermitian fundamental form $g_{ik} = g_{(ik)} + g_{[ik]}$ and an affine connection $\Gamma^i_{kl} = \Gamma^i_{(kl)} + \Gamma^i_{[kl]}$ which is Hermitian with respect to the lower indices.

Then the field equations of the Hermitian theory of relativity can be written as

$$g_{ik,l} - g_{nk}\Gamma^n_{il} - g_{in}\Gamma^n_{lk} = 0, \tag{1}$$

$$(\sqrt{-g}g^{[is]})_{,s} = 0, \tag{2}$$

$$R_{(ik)}(\Gamma) = 0, \tag{3}$$

$$R_{[ij],k}(\Gamma) + R_{[ki],j}(\Gamma) + R_{[jk],i}(\Gamma) = 0, \tag{4}$$

where $g = \det(g_{ik})$ and $R_{ik}(\Gamma)$ is the Ricci tensor

$$R_{ik}(\Gamma) = \Gamma^a_{ik,a} - \Gamma^a_{ia,k} - \Gamma^a_{ib}\Gamma^b_{ak} + \Gamma^a_{ik}\Gamma^b_{ab}. \tag{5}$$

Consider now a real symmetric tensor h_{ik} corresponding to a solution of the field equations of general relativity, which depends on the first three co-ordinates x^λ and for which $h_{\lambda 4} = 0$; we assume henceforth that Greek indices run from 1 to 3, while Latin indices run from 1 to 4. Consider also an antisymmetric purely imaginary tensor a_{ik} which depends on the first three co-ordinates; assume that its only nonvanishing components are $a_{\mu 4} = -a_{4\mu}$. Then form the mixed tensor

$$\alpha_i{}^k = a_{il}h^{lk} = -\alpha^k{}_i, \tag{6}$$

where h^{ik} is the inverse of h_{ik}, and define the Hermitian fundamental form g_{ik} as follows:

$$\begin{aligned} g_{\lambda\mu} &= h_{\lambda\mu}, \\ g_{4\mu} &= \alpha_4{}^\nu h_{\nu\mu}, \\ g_{44} &= h_{44} - \alpha_4{}^\mu \alpha_4{}^\nu h_{\mu\nu}. \end{aligned} \tag{7}$$

When the three additional conditions

$$\alpha^4{}_{\mu,\lambda} - \alpha^4{}_{\lambda,\mu} = 0 \tag{8}$$

are fulfilled, the affine connection Γ^i_{kl} which solves Eqs. (1) has the nonzero components

$$\begin{aligned} \Gamma^\lambda_{(\mu\nu)} &= \left\{ {\lambda \atop \mu\ \nu} \right\}, \\ \Gamma^\lambda_{[4\nu]} &= \alpha_4{}^\lambda{}_{,\nu} - \left\{ {4 \atop 4\ \nu} \right\} \alpha_4{}^\lambda + \left\{ {\lambda \atop \rho\ \nu} \right\} \alpha_4{}^\varrho, \\ \Gamma^4_{(4\nu)} &= \left\{ {4 \atop 4\ \nu} \right\}, \\ \Gamma^\lambda_{44} &= \left\{ {\lambda \atop 4\ 4} \right\} - \alpha_4{}^\nu \left(\Gamma^\lambda_{[4\nu]} - \alpha_4{}^\lambda \Gamma^4_{(4\nu)} \right); \end{aligned} \tag{9}$$

we indicate with $\left\{ {i \atop k\ l} \right\}$ the Christoffel connection built with h_{ik}; $\Gamma^\lambda_{[4\nu]}$ is just written as the covariant derivative of $\alpha_4{}^\lambda$ calculated with that connection.

We form now the Ricci tensor $R_{ik}(\Gamma)$. When Eqs. (2), i.e., in our case, the single equation

$$(\sqrt{-h}\ \alpha_4{}^{\lambda} h^{44})_{,\lambda} = 0, \tag{10}$$

and the additional conditions, expressed by Eqs. (8), are satisfied, the components of $R_{ik}(\Gamma)$ can be written as

$$\begin{aligned} R_{\lambda\mu} &= S_{\lambda\mu}, \\ R_{4\mu} &= \alpha_4{}^{\nu} S_{\nu\mu} + \left(\alpha_4{}^{\nu} \left\{ {4 \atop 4\ \nu} \right\}\right)_{,\mu}, \\ R_{44} &= S_{44} - \alpha_4{}^{\mu}\alpha_4{}^{\nu} S_{\mu\nu}, \end{aligned} \tag{11}$$

where S_{ik} is the Ricci tensor built with $\left\{ {i \atop k\ l} \right\}$. S_{ik} is zero when h_{ik} is a solution of the field equations of general relativity, as supposed; therefore, when Eqs. (8) and (10) hold, the Ricci tensor, defined by Eqs. (11), satisfies Eqs. (3) and (4) of the Hermitian theory of relativity.

3. Conclusion

We can resume our result as follows: let h_{ik} be the metric tensor for a solution of the field equations of general relativity which does not depend on, say, the fourth co-ordinate, and for which $h_{\lambda 4} = 0$. Let a_{ik} be an antisymmetric, pure imaginary tensor, which depends on the first three co-ordinates; $a_{4\mu} = -a_{\mu 4}$ are its only nonvanishing components. When a_{ik} obeys the field equation (10) and the additional conditions (8), the fundamental form g_{ik}, defined by Eqs. (7), provides a solution to the field equations of the Hermitian theory of relativity.

The task of solving Eqs. (1)-(4) reduces, under the circumstances considered here, to the simpler task of solving Eqs. (8) and (10) for a given h_{ik}; the particular solutions mentioned in the Introduction are physically meaningful examples of solutions built in this way.

We notice finally that the method proposed here applies also to Schrödinger's purely affine theory [6].

Aknowledgements. I am indebted to Prof. E. Kreisel and to Prof. H.-J. Treder for helpful discussions and for continuous support.

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