

A RULE FOR THE EQUILIBRIUM OF FORCES IN THE HERMITIAN THEORY OF RELATIVITY

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ABSTRACT. When the behaviour of the singularities, which are used to represent masses, charges or currents in exact solutions to the field equations of the Hermitian theory of relativity, is restricted by a no-jump rule, conditions are obtained, which determine the relative positions of masses, charges and currents. Due to these conditions the Hermitian theory of relativity appears to provide a unified description of gravitational, colour and electromagnetic forces.

Eine Regel für das Gleichgewicht der Kräfte in der Hermiteschen Relativitätstheorie

Inhaltsübersicht. Wenn das Verhalten der Singularitäten, die als Repräsentanten von Massen, Ladungen und Strömen in exacten Lösungen der Feldgleichungen der Hermiteschen Relativitätstheorie benutzt werden, durch eine Stetigkeitsforderung beschränkt wird, erhält man Bedingungen für die relativen Positionen von Massen, Ladungen und Strömen. Entsprechend diesen Bedingungen liefert die Hermitesche Relativitätstheorie eine vereinheitlichte Beschreibung der gravitativen, elektromagnetischen Kräfte und Farbkräfte.

1. INTRODUCTION

The Hermitian extension of the theory of general relativity, based on a Hermitian fundamental tensor $g_{ik} = g_{(ik)} + g_{[ik]}$, was conceived by Einstein in order to provide a unified description of gravitation and electrodynamics [1]. The study of solutions of the field equations has shown that the theory accounts for the existence of gravitational forces, as well as of forces which do not depend on the distance between charges which cannot exist as individuals [2, 3].

In the present paper, a rule for the regularity of the exact solutions is proposed, which confirms the previous findings and shows that, when the imaginary part $g_{[ik]}$ of the fundamental tensor fulfills Maxwell's equations, the forces of electrodynamics between charges and between currents are present too.

2. A RESTRICTIVE RULE FOR SINGULARITIES

The field equations of the Hermitian theory of relativity are [1]

$$\begin{aligned}
 (1) \quad & g_{ik,l} - g_{nk}\Gamma_{il}^n - g_{in}\Gamma_{lk}^n = 0, \\
 (2) \quad & (\sqrt{-g}g^{[is]}),_s = 0, \\
 (3) \quad & R_{(ik)}(\Gamma) = 0, \\
 (4) \quad & R_{[ij],k}(\Gamma) + R_{[ki],j}(\Gamma) + R_{[jk],i}(\Gamma) = 0,
 \end{aligned}$$

where g_{ik} is the previously mentioned fundamental tensor, and Γ_{kl}^i is an affine connection, which is Hermitian with respect to the lower indices, while $R_{ik}(\Gamma)$ is the Ricci tensor

$$(5) \quad R_{ik} = \Gamma_{ik,a}^a - \Gamma_{ia,k}^a - \Gamma_{ib}^a \Gamma_{ak}^b + \Gamma_{ik}^a \Gamma_{ab}^b;$$

a comma indicates ordinary differentiation.

Solutions of the field equations depending on two and three co-ordinates are known; when singularities are allowed in Eqs. (3) and (4), some of them appear to represent the static field of particles endowed with masses and with colour charges [3]. These solutions do not result to be singular only at the positions where the particles are located; they display also a lack of elementary flatness on lines stretching between the particles; the latter singularities disappear for such configurations of masses and charges, that one is led to infer that the Hermitian theory of relativity properly accounts for the existence of gravitational and colour forces.

When singularities in Eqs. (2) are allowed, the field equations of the theory admit solutions in which $g_{[ik]}$ fulfills Maxwell's equations, that clearly represent the electromagnetic fields associated with point charges, or with currents running on wires. In this case, however, no extra singularities in the expected form of deviations from elementary flatness are encountered, and the conclusion was drawn [3] that the electromagnetic field is there, but it is dynamically inactive. This conclusion is however unwarranted, since a more stringent regularity criterion exists, and its application to the known solutions, while confirming the previous results on the gravitational and colour forces, shows that the Hermitian theory of relativity accounts also for the electromagnetic forces.

The rule is the following: imagine that a solution is given, in which g_{ik} displays n point - or line - singularities, that we intend to interpret as masses, charges or currents, located at different positions in ordinary space. We consider such an interpretation as allowed only when each of these singularities does not contain a contribution, in the form of a jump, either finite or infinite, arising from the presence of the other masses, charges or currents. Obviously, such a rule has no pretense of rigour and generality; we consider it just as a heuristic tool, with which we want to investigate the known solutions.

3. CONSEQUENCES OF THE RESTRICTIVE RULE

First of all, let us see what happens when the restrictive criterion proposed above is applied to a Curzon solution [4] for n masses. Written in canonical cylindrical co-ordinates $x^1 = r$, $x^2 = z$, $x^3 = \varphi$, $x^4 = ct$, g_{ik} takes the Weyl-Levi Civita form [5]

$$(6) \quad g_{ik} = \begin{pmatrix} -\exp[2(\nu - \lambda)] & 0 & 0 & 0 \\ 0 & -\exp[2(\nu - \lambda)] & 0 & 0 \\ 0 & 0 & -r^2 \exp[-2\lambda] & 0 \\ 0 & 0 & 0 & \exp[2\lambda] \end{pmatrix},$$

where

$$(7) \quad \lambda = -\sum_{q=1}^n \frac{m_q}{p_q},$$

$$\nu = -\sum_{q=1}^n \frac{m_q^2 r^2}{2p_q^4} + \sum_{q,q'=1}^{n(q \neq q')} \frac{m_q m_{q'}}{(z_q - z_{q'})^2} \left(\frac{r^2 + (z - z_q)(z - z_{q'})}{p_q p_{q'}} - 1 \right);$$

m_q and z_q are constants, while

$$(8) \quad p_q = [r^2 + (z - z_q)^2]^{1/2}.$$

Such a solution presents singularities, which we intend to interpret as masses, at $r = 0$, $z = z_q$, but it does not satisfy the restrictive rule proposed above, due to the presence in ν of terms like

$$(9) \quad \mathcal{D}_q = m_q \sum_{q' \neq q}^n \frac{m_{q'}}{(z_q - z_{q'})^2} \frac{r^2 + (z - z_q)(z - z_{q'})}{p_q p_{q'}}.$$

Imagine that we cross the q -th singularity along a given line; when the co-ordinates of the running point differ very few from the co-ordinates of the q -th singularity we have

$$(10) \quad \mathcal{D}_q \cong m_q \frac{z - z_q}{p_q} \sum_{q' \neq q}^n m_{q'} \frac{z_q - z_{q'}}{|z_q - z_{q'}|^3}.$$

Therefore the term \mathcal{D}_q , and hence g_{ik} , presents a finite jump at the position of the q -th singularity, unless

$$(11) \quad \sum_{q' \neq q}^n m_{q'} \frac{z_q - z_{q'}}{|z_q - z_{q'}|^3} = 0.$$

When Eq. (11) holds for any value of q , i.e. when the restrictive criterion is fulfilled, even the requirement of elementary flatness on the z -axis is satisfied, and vice-versa; the no-jump rule, applied to the Curzon solution, provides the relativistic version of the condition for the equilibrium of n collinear masses at rest.

Let us consider now that solution of the field equations of the Hermitian theory for which the fundamental tensor, again referred to cylindrical co-ordinates r, z, φ, ct , reads

$$(12) \quad g_{ik} = \begin{pmatrix} -1 & 0 & \delta & 0 \\ 0 & -1 & \varepsilon & 0 \\ -\delta & -\varepsilon & \zeta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

with

$$(13) \quad \delta = i \sum_{q=1}^n \frac{k_q r (z - z_q)}{p_q}, \quad \varepsilon = -i \sum_{q=1}^n \frac{k_q r^2}{p_q},$$

$$\zeta = -r^2 \left(1 + \sum_{q=1}^n k_q^2 + \sum_{q, q'=1}^{n(q \neq q')} k_q k_{q'} \frac{r^2 + (z - z_q)(z - z_{q'})}{p_q p_{q'}} \right);$$

k_q and z_q are constants, while p_q is again defined by Eq. (8), and $i = \sqrt{-1}$. When the sum rule

$$(14) \quad \sum_{q=1}^n k_q = 0$$

is satisfied, the solution, considered in Cartesian co-ordinates, tends to Minkowski values at space infinity. It displays singularities at $r = 0, z = z_q$, which we intend to interpret [2, 3] as colour charges, provided that the no-jump condition is satisfied at the singularities. This occurrence is not verified in general, due to the presence in ζ of terms like

$$(15) \quad \mathcal{D}_q = k_q \sum_{q' \neq q}^n k_{q'} \frac{r^2 + (z - z_q)(z - z_{q'})}{p_q p_{q'}};$$

by repeating here the argument used with the Curzon solution, we find that the no-jump condition is satisfied if

$$(16) \quad \sum_{q' \neq q}^n k_{q'} \frac{z_q - z_{q'}}{|z_q - z_{q'}|} = 0.$$

When Eq. (16) holds for all the values of q , elementary flatness can be ensured on the whole z -axis, and vice-versa; Eq. (16) expresses the equilibrium condition of n colour charges k_q at rest, mutually interacting with forces which do not depend on the distance [2, 3].

What happens now, when the restrictive condition on singularities is applied to an electrostatic solution of the field equations of the Hermitian theory? The solution, written in Cartesian co-ordinates $x^1 = x, x^2 = y,$

$x^3 = z$, $x^4 = ct$, reads [3]

$$(17) \quad g_{ik} = \begin{pmatrix} -1 & 0 & 0 & a \\ 0 & -1 & 0 & b \\ 0 & 0 & -1 & c \\ -a & -b & -c & d \end{pmatrix},$$

with

$$(18) \quad a = i \sum_{q=1}^n \frac{h_q(x-x_q)}{p_q^3}, \quad b = i \sum_{q=1}^n \frac{h_q(y-y_q)}{p_q^3}, \quad c = i \sum_{q=1}^n \frac{h_q(z-z_q)}{p_q^3},$$

$$d = 1 - \sum_{q=1}^n \frac{h_q^2}{p_q^4}$$

$$- \sum_{q,q'=1}^{n(q \neq q')} h_q h_{q'} \frac{(x-x_q)(x-x_{q'}) + (y-y_q)(y-y_{q'}) + (z-z_q)(z-z_{q'})}{p_q^3 p_{q'}^3};$$

now

$$(19) \quad p_q = [(x-x_q)^2 + (y-y_q)^2 + (z-z_q)^2]^{1/2},$$

while h_q , x_q , y_q and z_q are constants. We would like to interpret the singularities of this solution, occurring at $x = x_q$, $y = y_q$, $z = z_q$, as point electric charges, but we are prevented to do always so by the restrictive rule, which is not fulfilled in general, due to the occurrence in d of terms like

$$(20) \quad \mathcal{D}_q = h_q \sum_{q' \neq q}^n h_{q'} \frac{(x-x_q)(x-x_{q'}) + (y-y_q)(y-y_{q'}) + (z-z_q)(z-z_{q'})}{p_q^3 p_{q'}^3}.$$

If we define

$$(21) \quad r_{qq'} = [(x_q - x_{q'})^2 + (y_q - y_{q'})^2 + (z_q - z_{q'})^2]^{1/2},$$

we find, in the neighbourhood of the q -th singularity

$$(22) \quad \mathcal{D}_q \cong h_q \sum_{q' \neq q}^n h_{q'} \frac{(x-x_q)(x_q-x_{q'}) + (y-y_q)(y_q-y_{q'}) + (z-z_q)(z_q-z_{q'})}{p_q^3 r_{qq'}^3}.$$

The no-jump condition then requires

$$(23) \quad \sum_{q' \neq q}^n h_{q'} \frac{x_q - x_{q'}}{r_{qq'}^3} = \sum_{q' \neq q}^n h_{q'} \frac{y_q - y_{q'}}{r_{qq'}^3} = \sum_{q' \neq q}^n h_{q'} \frac{z_q - z_{q'}}{r_{qq'}^3} = 0$$

for any value of q . Eqs. (23) are just the equilibrium conditions for n point charges h_q at rest, according to electrostatics.

We now look at the electromagnetic solution [3] for which the fundamental tensor, expressed in Cartesian co-ordinates x, y, z, ct , reads

$$(24) \quad g_{ik} = \begin{pmatrix} -1 & 0 & e & 0 \\ 0 & -1 & f & 0 \\ -e & -f & h & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

with

$$(25) \quad e = i \sum_{q=1}^n \frac{l_q(x-x_q)}{p_q^2}, \quad f = i \sum_{q=1}^n \frac{l_q(y-y_q)}{p_q^2},$$

$$h = -1 - \sum_{q=1}^n \frac{l_q^2}{p_q^2} - \sum_{q,q'=1}^{n(q \neq q')} l_q l_{q'} \frac{(x-x_q)(x-x_{q'}) + (y-y_q)(y-y_{q'})}{p_q^2 p_{q'}^2},$$

where now

$$(26) \quad p_q = [(x-x_q)^2 + (y-y_q)^2]^{1/2},$$

and l_q, x_q, y_q are arbitrary constants. We intend to interpret this solution as describing the magnetic field due to n wires parallel to the z axis, on which steady currents l_q are running, but we can not do so in general, due to the occurrence in h of terms like

$$(27) \quad \mathcal{D}_q = l_q \sum_{q' \neq q}^n l_{q'} \frac{(x-x_q)(x-x_{q'}) + (y-y_q)(y-y_{q'})}{p_q^2 p_{q'}^2}.$$

Let us set

$$(28) \quad d_{qq'} = [(x_q - x_{q'})^2 + (y_q - y_{q'})^2]^{1/2};$$

then, in the neighbourhood of the q -th wire we have

$$(29) \quad \mathcal{D}_q \cong l_q \sum_{q' \neq q}^n l_{q'} \frac{(x-x_q)(x_q - x_{q'}) + (y-y_q)(y_q - y_{q'})}{p_q^2 d_{qq'}^2}.$$

\mathcal{D}_q , and hence g_{ik} , displays an infinite jump when the q -th wire is crossed, unless

$$(30) \quad \sum_{q' \neq q}^n l_{q'} \frac{x_q - x_{q'}}{d_{qq'}^2} = \sum_{q' \neq q}^n l_{q'} \frac{y_q - y_{q'}}{d_{qq'}^2} = 0$$

for any value of q . Eqs. (30) are just the equilibrium conditions for n parallel wires at rest, run by steady currents l_q , according to electrodynamics.

4. CONCLUSION

The exact solutions and the restrictive rule for singularities considered in this paper suggest that Einstein's Hermitian theory of relativity provides a unified description of gravodynamics, chromodynamics and electrodynamics.

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