THE ORIGIN OF THE ELECTROMAGNETIC INTERACTION IN EINSTEIN'S UNIFIED FIELD THEORY WITH SOURCES

S. ANTOCI

ABSTRACT. Einstein's unified field theory is extended by the addition of matter terms in the form of a symmetric energy tensor and of two conserved currents. From the field equations and from the conservation identities emerges the picture of a gravoelectrodynamics in a dynamically polarizable Riemannian continuum. Through an approximate calculation exploiting this dynamical polarizability it is argued that ordinary electromagnetism may be contained in the theory.

1. INTRODUCTION

Recently, it has been shown that, if sources are appended in a certain way to the field equations of Einstein's unified field theory [1], the contracted Bianchi identities and the field equations appear endowed with definite physical meaning. The theory looks like a gravoelectrodynamics in a polarizable Riemannian continuum [2], in which the relationship between the electromagnetic inductions and fields is dictated by the field equations in a way that deserves a thorough scrutiny, for the wealth of the implied possibilities is far richer than in the so-called Einstein-Maxwell theory. As a partial contribution to the understanding of these new opportunities, we show here a way by which the particular occurrence that we consider ordinary electrodynamic interaction stems from the theory.

2. EINSTEIN'S GRAVOELECTRODYNAMICS WITH SOURCES

On a four-dimensional, real manifold, let $\mathbf{g}^{ik}$ be a contravariant tensor density with an even part $\mathbf{g}^{(ik)}$ and an alternating one $\mathbf{g}^{[ik]}$:

$$(1) \qquad \mathbf{g}^{ik} = \mathbf{g}^{(ik)} + \mathbf{g}^{[ik]}.$$

We also endow the manifold with a general affine connection

$$(2) \qquad W^i_{kl} = W^i_{(kl)} + W^i_{[kl]}.$$

Out of this affinity we form the Riemann curvature tensor

$$(3) \qquad R^i{}_{klm}(W) = W^i_{kl,m} - W^i_{km,l} - W^i_{al}W^a_{km} + W^i_{am}W^a_{kl}$$

for which two distinct contractions, $R_{ik}(W) = R^p{}_{ikp}(W)$ and $A_{ik}(W) = R^p{}_{pik}(W)$ exist [3]. But even the transposed affinity $\tilde{W}^i_{kl} = W^i_{lk}$ shall be taken into account: from it, the Riemann curvature tensor $R^i{}_{klm}(\tilde{W})$ and its two contractions $R_{ik}(\tilde{W})$ and $A_{ik}(\tilde{W})$ can be formed as well. We aim to follow the pattern of general relativity, in which the Lagrangian density $\mathbf{g}^{ik}R_{ik}$ is considered, but now any linear combination $\bar{R}_{ik}$ of the four above-mentioned contractions can be envisaged. A good choice [4], for reasons that will become apparent later, is

$$(4) \qquad\qquad \bar{R}_{ik}(W) = R_{ik}(W) + \frac{1}{2}A_{ik}(\tilde{W}).$$

Since we do not believe that we will attain a complete theory, i.e. one in which the energy tensor and the currents that describe matter are determined by the field equations, we endow the theory with sources in the form of a nonsymmetric tensor P_{ik} and of a current density $\mathbf{j}^i$, and we couple them to $\mathbf{g}^{ik}$ and to the vector $W_i = W^l_{[il]}$ respectively. The Lagrangian density

$$(5) \qquad\qquad \mathbf{L} = \mathbf{g}^{ik}\bar{R}_{ik}(W) - 8\pi\mathbf{g}^{ik}P_{ik} + \frac{8\pi}{3}W_i\mathbf{j}^i$$

is thus arrived at. By performing independent variations of the action $\int \mathbf{L}d\Omega$ with respect to W^p_{qr} and to $\mathbf{g}^{ik}$ with suitable boundary conditions we obtain the field equations

$$(6) \qquad\qquad -\mathbf{g}^{qr}_{,p} + \delta^r_p\mathbf{g}^{(sq)}_{,s} - \mathbf{g}^{sr}W^q_{sp} - \mathbf{g}^{qs}W^r_{ps}$$
$$+\delta^r_p\mathbf{g}^{st}W^q_{st} + \mathbf{g}^{qr}W^t_{pt} = \frac{4\pi}{3}(\mathbf{j}^r\delta^q_p - \mathbf{j}^q\delta^r_p)$$

and

$$(7) \qquad\qquad \bar{R}_{ik}(W) = 8\pi P_{ik}.$$

By contracting eq. (6) with respect to q and p we get

$$(8) \qquad\qquad \mathbf{g}^{[is]}_{,s} = 4\pi\mathbf{j}^i,$$

a desirable outcome. We get a problem too, since the very existence of the latter equation means that we cannot determine the affinity W^i_{kl} uniquely in terms of $\mathbf{g}^{ik}$: eq. (6) is in fact invariant under the projective transformation $W'^i_{kl} = W^i_{kl} + \delta^i_k\lambda_l$, with λ_l arbitrary vector field. And moreover eq. (7) is invariant under the transformation

$$(9) \qquad\qquad W'^i_{kl} = W^i_{kl} + \delta^i_k\mu_{,l}$$

where μ is an arbitrary scalar. Equation (8) and the invariance under (9) are hints for a possible electromagnetic content of the theory. We can write

$$(10) \qquad\qquad W^i_{kl} = \Gamma^i_{kl} - \frac{2}{3}\delta^i_k W_l$$

where Γ^i_{kl} is another affine connection, by definition constrained to yield $\Gamma^l_{[il]=0}$. Then eq. (6) becomes

(11) $$\mathbf{g}^{qr}_{,p} + \mathbf{g}^{sr}\Gamma^q_{sp} + \mathbf{g}^{qs}\Gamma^r_{ps} - \mathbf{g}^{qr}\Gamma^t_{(pt)} = \frac{4\pi}{3}(\mathbf{j}^q\delta^r_p - \mathbf{j}^r\delta^q_p)$$

and allows us to determine Γ^i_{kl} uniquely, under very general conditions, in terms of $\mathbf{g}^{ik}$. When eq. (10) is substituted in eq. (7), the latter comes to read

(12) $$\bar{R}_{(ik)}(\Gamma) = 8\pi P_{(ik)}$$

(13) $$\bar{R}_{[ik]}(\Gamma) = 8\pi P_{[ik]} - \frac{1}{3}(W_{i,k} - W_{k,i})$$

after splitting the even and the alternating parts. Wherever the source term is nonvanishing, a field equation loses its meaning, and becomes a definition of some property of matter in terms of geometrical entities; it is quite obvious that such a definition must be unique. This occurs with eqs. (8), (11) and (12), but it does not happen for eq. (13). This equation only prescribes that $\bar{R}_{[ik]}(\Gamma) - 8\pi P_{[ik]}$ is the curl of the arbitrary vector $W_i/3$; it is equivalent to the four equations

(14) $$\bar{R}_{[[ik],l]}(\Gamma) = 8\pi P_{[[ik],l]}$$

and cannot specify $P_{[ik]}$ uniquely. We therefore scrap the redundant tensor $P_{[ik]}$, as we scrapped the redundant affinity W^i_{kl}, and assume henceforth that matter is defined by the symmetric tensor $P_{(ik)}$, by the current density $\mathbf{j}^i$ and by the current

(15) $$K_{ikl} = \frac{1}{8\pi}\bar{R}_{[[ik],l]},$$

both $\mathbf{j}^i$ and K_{ikl} being conserved quantities by definition. The analogy with general relativity, to which the present theory formally reduces when $\mathbf{g}^{[ik]} = 0$, suggests rewriting eq. (12) as

(16) $$\bar{R}_{(ik)}(\Gamma) = 8\pi(T_{ik} - \frac{1}{2}s_{ik}s^{pq}T_{pq})$$

where $s_{ik} = s_{ki}$ is the still unchosen metric tensor of the theory, $s^{il}s_{kl} = \delta^i_k$, and the symmetric tensor T_{ik} will act as energy tensor.

Equations (11), (16), (8) and (15) reduce to the equations of Einstein's unified field theory when sources are absent, since then $\bar{R}_{ik}(\Gamma)=R_{ik}(\Gamma)$; moreover they enjoy the property of transposition invariance even when sources are present. If $\mathbf{g}^{ik}$, Γ^i_{kl}, $\bar{R}_{ik}(\Gamma)$ represent a solution with the sources T_{ik}, $\mathbf{j}^i$ and K_{ikl}, the transposed quantities $\tilde{\mathbf{g}}^{ik} = \mathbf{g}^{ki}$, $\tilde{\Gamma}^i_{kl} = \Gamma^i_{lk}$ and $\bar{R}_{ik}(\tilde{\Gamma})=\bar{R}_{ki}(\Gamma)$ represent another solution, endowed with the sources $\tilde{T}^{ik} = T_{ik}, \tilde{\mathbf{j}}^i = -\mathbf{j}^i$ and $\tilde{K}_{ikl} = -K_{ikl}$. Such a desirable property is a consequence of the choice made for $\bar{R}_{ik}$. These equations suggest interpreting Einstein's unified field theory with sources as a gravoelectrodynamics in a polarizable continuum, allowing for both electric and magnetic currents. The study of the

conservation identities confirms the idea and provides at the same time the identification of the metric tensor s_{ik}. Let us consider the invariant integral

$$(17) \qquad I = \int \left[\mathbf{g}^{ik} \bar{R}_{ik}(W) + \frac{8\pi}{3} W_i \mathbf{j}^i \right] d\Omega.$$

From it, when eq. (6) is assumed to hold, by means of an infinitesimal coordinate transformation we get the four identities

$$(18) \qquad -(\mathbf{g}^{is}\bar{R}_{ik}(W) + \mathbf{g}^{si}\bar{R}_{ki}(W))_{,s} + \mathbf{g}^{pq}\bar{R}_{pq,k}(W)$$
$$+\frac{8\pi}{3}\mathbf{j}^i(W_{i,k} - W_{k,i}) = 0.$$

This equation can be rewritten as

$$(19) \qquad -2(\mathbf{g}^{(is)}\bar{R}_{(ik)}(\Gamma))_{,s} + \mathbf{g}^{(pq)}\bar{R}_{(pq),k}(\Gamma)$$
$$= 2\mathbf{g}^{[is]}_{,s}\bar{R}_{[ik]}(\Gamma) + \mathbf{g}^{[is]}\bar{R}_{[[ik],s]}(\Gamma)$$

where the redundant variable W^i_{kl} no longer appears. Let us remember eq. (16) and assume that the metric tensor is defined by the equation [5]

$$(20) \qquad \sqrt{-s}s^{ik} = \mathbf{g}^{(ik)}$$

where $s = \det(s_{ik})$; we shall use henceforth s^{ik} and s_{ik} to raise and lower indices, $\sqrt{-s}$ to produce tensor densities out of tensors. We define then

$$(21) \qquad \mathbf{T}^{ik} = \sqrt{-s}s^{ip}s^{kq}T_{pq}$$

and the weak identities (19), when all the field equations hold, will take the form

$$(22) \qquad \mathbf{T}^{ls}_{;s} = \frac{1}{2}s^{lk}(\mathbf{j}^i\bar{R}_{[ki]}(\Gamma) + K_{iks}\mathbf{g}^{[si]})$$

where the semicolon indicates the covariant derivative with respect to the Christoffel affinity

$$(23) \qquad \left\{ {}^{\,i}_{k\,l} \right\} = \frac{1}{2}s^{im}(s_{mk,l} + s_{ml,k} - s_{kl,m})$$

built with s_{ik}. Our earlier impression is confirmed by eq. (22): the theory, built in terms of a non-Riemannian geometry, entails a gravoelectrodynamics in a dynamically polarized Riemannian spacetime, for which s_{ik} is the metric. The relationship between electromagnetic inductions and fields is governed by the field equations in a quite novel and subtle way, with respect to the one prevailing in the so-called Einstein-Maxwell theory. Two versions of Einstein's gravoelectrodynamics are possible, according to whether $\mathbf{g}^{ik}$ is chosen to be a real nonsymmetric or a complex Hermitian tensor density.

3. The origin of the electromagnetic interaction

In the present theory, finding exact solutions to the field equations not restricted by symmetry or by some other limitation is by no means easy; even the straightforward task of writing the sources explicitly in terms of a general $\mathbf{g}^{ik}$ leads to unsurveyable expressions. We therefore bow to the need to perform approximate calculations, and assume that, while $\mathbf{s}^{ik}$ is an arbitrary tensor density, $\mathbf{g}^{[ik]}$, when compared to $\mathbf{s}^{ik}$, is a small, first order quantity, that we call henceforth $\mathbf{a}^{ik}$. As previously noted, we raise and lower indices with s^{ik} and s_{ik}, build tensor densities with $\sqrt{-s}$, etc. Thus we have

$$(24) \qquad a_i{}^k = s_{il}\mathbf{a}^{lk}/\sqrt{-s} = -a^k{}_i$$

$$a_{ik} = a_i{}^l s_{kl} = -a_{ki}$$

$$j_i = (1/4\pi)a_i{}^s{}_{;s}.$$

The structure of the field equations and of the conservation identities is such that a consistent approximation scheme is attained if we calculate the alternating quantities in first order, the even quantities up to second order in a_{ik}. Due to the invariance under transposition, a_{ik} and its derivatives will appear in the even quantities only through second order combinations. With this proviso, we solve eq. (11) for Γ^i_{kl} up to second order; the approximate solution reads

$$(25) \qquad \Gamma^i_{kl} = \left\{ {}^i_{k\,l} \right\} + \Theta^i_{kl} + \Sigma^i_{kl}$$

where the Christoffel symbol of eq. (23) is the zeroth order contribution, while $\Theta^i_{kl} = -\Theta^i_{lk}$ is the first order part, and $\Sigma^i_{kl} = \Sigma^i_{lk}$ is the second order correction. We get

$$(26) \qquad \Theta^i_{kl} = \frac{1}{2}s^{im}(a_{km;l} + a_{ml;k} - a_{lk;m}) + \frac{4\pi}{3}(\delta^i_k j_l - \delta^i_l j_k)$$

and

$$(27) \qquad \Sigma^i_{kl} = \frac{1}{2}\left\{ a^{si}(a_{ks;l} + a_{ls;k}) + a_k{}^s a^i{}_{l;s} + a_l{}^s a^i{}_{k;s} \right\}$$

$$+\frac{1}{4}a^{ms}(\delta^i_k a_{ms;l} + \delta^i_l a_{ms;k} - s^{ip}s_{kl}a_{ms;p})$$

$$+\frac{2\pi}{3}j_s(\delta^i_l a_k{}^s + \delta^i_k a_l{}^s - 3s_{kl}a^{is}).$$

Let us call $S^i{}_{klm}$ the Riemann tensor built with $\left\{ {}^i_{k\,l} \right\}$, S_{ik} the corresponding Ricci tensor. When $\bar{R}_{(ik)}(\Gamma)$ is calculated up to second order we find

$$(28) \qquad \bar{R}_{(ik)}(\Gamma) = S_{ik} + \Sigma^a_{ik;a} - \frac{1}{2}\left(\Sigma^a_{ia;k} + \Sigma^a_{ka;i} \right) - \Theta^a_{ib}\Theta^b_{ak}$$

$$(29) \qquad\qquad\qquad\qquad \bar{R}_{[ik]}(\Gamma) = \Theta^a_{ik;a}$$

for the even and the alternating parts respectively. We wish to understand how the field equations rule the relationship between $\mathbf{a}^{ik}$ and $\bar{R}_{[ik]}$, and what sort of interactions arise from the right-hand side of eq. (22) when $\mathbf{j}^i$ and K_{ikl}

are everywhere vanishing, except in preassigned world tubes. In particular, we wish to appreciate in what manner, if any, the usual electrodynamic interaction is an outcome of the theory. We need not require that the energy tensor T_{ik} be vanishing outside the world tubes; we know in advance from eq. (22) that, wherever the two currents annihilate, T_{ik} displays a pure gravito-inertial behaviour in the Riemannian spacetime for which s_{ik} is the metric. It will suffice to solve eqs. (8) and (15) with the sources $\mathbf{j}^i$ and K_{ikl} localized in the above sense. When written in terms of s_{ik} and a_{ik}, $\bar{R}_{[ik]}(\Gamma)$ turns out to be

$$(30) \qquad \bar{R}_{[ik]}(\Gamma) = \frac{2\pi}{3}(j_{i,k} - j_{k,i}) + \frac{1}{2}a_i{}^n S_{nk} - \frac{1}{2}a_k{}^n S_{ni}$$
$$-a^{pq}S_{pikq} + \frac{1}{2}s^{pq}a_{ik;p;q}.$$

Given the form of $\bar{R}_{[ik]}(\Gamma)$, a general solution of our problem seems beyond reach, but a particular solution, of immediate physical interest, is at hand. Let us assume that a_{ik} is a pure curl:

$$(31) \qquad a_{ik} = \phi_{k,i} - \phi_{i,k}$$

everywhere. Then a_{ik} has to obey both sets of Maxwell's equations in the Riemannian spacetime described by s_{ik}; the vector potential ϕ_i shall be so chosen as to ensure the fulfillment of the equation $a_i{}^s{}_{;s} = 0$ everywhere except for preassigned world tubes. Due to eq. (31) $\bar{R}_{[ik]}(\Gamma)$ then reads

$$(32) \qquad \bar{R}_{[ik]}(\Gamma) = \frac{8\pi}{3}(j_{i,k} - j_{k,i}) + a_i{}^n S_{nk} - a_k{}^n S_{ni} + a^{pq}S_{pqik}.$$

Remember now the definition of the conformal curvature tensor [6]

$$(33) \qquad C_{ijkl} = S_{ijkl} - \frac{1}{2}\left(S_{jk}s_{il} + s_{jk}S_{il} - S_{jl}s_{ik} - s_{jl}S_{ik}\right)$$
$$+ \frac{S}{6}\left(s_{jk}s_{il} - s_{jl}s_{ik}\right)$$

where $S = s^{ik}S_{ik}$ is the scalar curvature of $S^i{}_{klm}$; eq. (32) can be rewritten as

$$(34) \qquad \bar{R}_{[ik]}(\Gamma) = \frac{8\pi}{3}(j_{i,k} - j_{k,i}) + a^{pq}C_{pqik} + \frac{S}{3}a_{ik}.$$

Assume that $C_{iklm} = 0$ everywhere, i.e. that the Riemannian spacetime described by s_{ik} is conformally flat, and that S vanishes outside the world tubes. Since $\bar{R}_{[ik]}(\Gamma)$ is a first order quantity, these requirements need to be met in zeroth order only. Then, to the required order, $\bar{R}_{[ik]}(\Gamma)$ vanishes outside the world tubes, while inside it reads

$$(35) \qquad \bar{R}_{[ik]}(\Gamma) = \frac{8\pi}{3}(j_{i,k} - j_{k,i}) + \frac{S}{3}a_{ik}.$$

We imagine now that the world tubes are so chosen that the intersection of any one of them with an arbitrary spacelike hypersurface can be individually surrounded by a closed and otherwise arbitrary two-surface entirely

lying where $\bar{R}_{[ik]}(\Gamma) = 0$, and we consider what sort of interactions are dictated, under these conditions, by the right-hand side of eq. (22). We see that, although K_{ikl} is in general nonvanishing inside a world tube, Gauss theorem proves that this current cannot give rise to net charges, for $\bar{R}_{[ik]}(\Gamma)$ vanishes on the two-surface whose existence was supposed above; only multipole interactions can be expected from the second term at the right-hand side of eq. (22). From the first term we get two contributions: one is a current-current self-interaction that vanishes when the curl of j_i is zero; the other one, which we call $\mathbf{f}^l$, reads

$$(36) \qquad\qquad \mathbf{f}^l = \frac{S}{6}\mathbf{j}_i a^{li}.$$

Since a_{ik} obeys both sets of Maxwell's equations, $\mathbf{f}^l$ is a Lorentz force density acting on the current Sj_i. If we assume that the gradient of S is orthogonal to j^i, so that the scalar curvature has constant value along a streamline of j^i, the correspondence with the ordinary electrodynamic interaction becomes complete, since the current Sj_i is then a conserved quantity that can build conserved net charges, like j_i does. The sign of the interaction can be adjusted to meet the experimental evidence both in the real nonsymmetric and in the complex Hermitian versions of the theory by properly choosing the sign of S.

4. Concluding remarks

We have provided, through an approximate calculation, evidence of how the usual electrodynamic interaction can stem from Einstein's gravoelectrodynamics. We feel confident, after this result, in identifying $\mathbf{j}^i$ with the electric current density and K_{ikl} with the magnetic current, $\mathbf{g}^{[ik]}$ with the electric induction and the magnetic field, $\bar{R}_{[ik]}(\Gamma)$ with the electric field and the magnetic induction. Due to the polarizability properties of the continuum it is not required that, in order to produce the electrodynamic interaction, a propagating $\bar{R}_{[ik]}(\Gamma)$ should correspond to each propagating $\mathbf{g}^{[ik]}$. A charged particle with the appropriate geometrical structure can influence the polarizability and induce the interaction with its mere presence.

One should not think, however, that this is the only way the electrodynamic interaction can arise from the theory: as can be readily appreciated from eq. (30), if $a_{[ik,l]} \neq 0$ and $S^i{}_{klm} = 0$ everywhere, the field equations ensure, where currents are absent, that $\bar{R}_{[ik]}(\Gamma)$ must fulfill the two equations

$$(37) \qquad\qquad (\sqrt{-s}\, s^{il} s^{km} \bar{R}_{[lm]}(\Gamma))_{,k} = 0$$

$$(38) \qquad\qquad \bar{R}_{[[ik],l]}(\Gamma) = 0.$$

Examples of this occurrence with a nonvanishing, propagating $\bar{R}_{[ik]}(\Gamma)$ have already been provided [7, 8]. Nor should one think that the scope of Einstein's gravoelectrodynamics is limited to accounting for Maxwell's electromagnetism and gravitation: just in the above-mentioned examples $\mathbf{g}^{[ik]}$

behaves in such a way that the last term of eq. (22) shows that magnetic charges built by K_{ikl} interact with forces not depending on distance. The interaction of unlike magnetic charges turns out to be attractive, hence confining, in the complex Hermitian version of the theory.

REFERENCES

1. Einstein, A. (1925). *S. B. Preuss. Akad. Wiss.*, **22**, 414; Einstein, A. (1945). *Ann. Math.*, **46**, 578; Einstein, A., and Straus, E.G. (1946). *Ann. Math.*, **47**, 731; Einstein, A. (1948). Rev. Mod. Phys., **20**, 35; Einstein, A., and Kaufman, B. (1955). *Ann. Math.*, **62**, 128.
2. Antoci, S. (1990). *Prog. Theor. Phys.*, **83**, 953.
3. Schrödinger, E. (1950). *Space-Time Structure*, (Cambridge University Press, Cambridge).
4. Borchsenius, K. (1978). *Nuovo Cimento*, **46A**, 403.
5. Hély, J. (1954). *Comptes Rend. Acad. Sci. (Paris)*, **239**, 385.
6. Goldberg, S. I. (1962). *Curvature and Homology*, (Academic Press, New York).
7. Treder, H. (1957). *Ann. Phys. (Leipzig)*, **19**, 369; *id.* (1980). *Ann. Phys. (Leipzig)*, **37**, 250.
8. Antoci, S. (1987). *Prog. Theor. Phys.*, **78**, 815.

Dipartimento di Fisica "A. Volta", via Bassi 6, 27100 Pavia, Italy