THE ELECTROSTATICS OF EINSTEIN'S UNIFIED FIELD THEORY

SALVATORE ANTOCI, DIERCK-EKKEHARD LIEBSCHER, AND LUIGI MIHICH

Abstract. When sources are added at their right-hand sides, and $g_{(ik)}$ is a priori assumed to be the metric, the equations of Einstein's Hermitian theory of relativity were shown to allow for an exact solution that describes the general electrostatic field of n point charges. Moreover, the injunction of spherical symmetry of $g_{(ik)}$ in the infinitesimal neighbourhood of each of the charges was proved to yield the equilibrium conditions of the n charges in keeping with ordinary electrostatics. The tensor $g_{(ik)}$, however, cannot be the metric of the theory, since it enters neither the eikonal equation nor the equation of motion of uncharged test particles. A physically correct metric that rules both the behaviour of wave fronts and of uncharged matter is the one indicated by Hély.

In the present paper it is shown how the electrostatic solution predicts the structure of the n charged particles and their mutual positions of electrostatic equilibrium when Hély's physically correct metric is adopted.

1. Introduction

The Hermitian theory of relativity [3] is nearly forgotten by the theoreticians of the present time; the few, who still have some remembrance of the efforts done both by Einstein [1, 2, 3, 4] and by Schrödinger [5, 6, 7] to find a generalization of the theory of 1915 that could encompass both gravitation and electromagnetism in a unique geometrical structure, consider the matter as a subject of purely historical interest.

One may observe, however, that no conclusive evidence was ever brought against the theory, either through the exact solutions [8, 9] or through approximate calculations [10]. Nevertheless, since no cogent identification of the geometrical objects of the theory with physical entities had been achieved, the interest aroused by the theory at the time of its appearance simply faded away with the lapse of the years.

A persistent prejudice, that has presumably undermined the proper understanding of the theory, has been the initial, tentative adoption, as metric tensor, of the symmetric part $g_{(ik)}$ of the fundamental tensor. Not everybody has incurred in such a prejudice; Lichnerowicz, for instance [11, 12], rightly based his choice of the metric on the eikonal equation of the theory; however his argument can determine that choice only up to an unknown conformal factor. By challenging Einstein's and Schrödinger's conviction, that the theory was a complete one, hence no source terms were needed at

the right-hand sides of the field equations, Hély [13, 14] had succeeded in giving a physically meaningful form to the conservation identities. He had obtained this result by adopting as metric a particular tensor s_{ik} that belongs to the class of conformally related metrics allowed for by the argument of Lichnerowicz. However, Hély's achievement went unnoticed, apart from a few exceptions, e.g. [15].

When the class of exact solutions depending on three coordinates and reported in [16] was found, the solutions were analysed by assuming $g_{(ik)}$ as metric. It is mandatory to reconsider all these exact solutions by adopting Hély's choice. We begin here, by reconsidering what already seemed, with $g_{(ik)}$ as metric, to be a sort of general electrostatic solution. We show that it fully deserves such a title when the physically correct metric is adopted.

2. The field equations.

A given geometric quantity [17] will be called hereafter Hermitian with respect to the indices i and k, both either covariant or contravariant, if the part of the quantity that is symmetric with respect to i and k is real, while the part that is antisymmetric is purely imaginary. By extending into the complex domain the symmetry postulates of general relativity, let us consider the Hermitian fundamental form $g_{ik} = g_{(ik)} + g_{[ik]}$ and the affine connection $\Gamma^i_{kl} = \Gamma^i_{(kl)} + \Gamma^i_{[kl]}$, Hermitian with respect to the lower indices; both entities depend on the real coordinates x^i, with i running from 1 to 4. We define also the Hermitian contravariant tensor g^{ik} by the relation

$$g^{il} g_{kl} = \delta^i_k, \tag{2.1}$$

and the contravariant tensor density $\mathbf{g}^{ik} = (-g)^{1/2} g^{ik}$, where $g \equiv \det(g_{ik})$ is a real quantity. Then the field equations of Einstein's unified field theory in the complex Hermitian form [3] read

$$g_{ik,l} - g_{nk}\Gamma^n_{il} - g_{in}\Gamma^n_{lk} = 0, \tag{2.2}$$

$$\mathbf{g}^{[is]}_{,s} = 0, \tag{2.3}$$

$$R_{(ik)}(\Gamma) = 0, \tag{2.4}$$

$$R_{[ik],l}(\Gamma) + R_{[kl],i}(\Gamma) + R_{[li],k}(\Gamma) = 0; \tag{2.5}$$

$R_{ik}(\Gamma)$ is the Ricci tensor

$$R_{ik}(\Gamma) = \Gamma^a_{ik,a} - \Gamma^a_{ia,k} - \Gamma^a_{ib}\Gamma^b_{ak} + \Gamma^a_{ik}\Gamma^b_{ab}. \tag{2.6}$$

3. The general electrostatic solution.

When referred to the coordinates $x^1 = x$, $x^2 = y$, $x^3 = z$, $x^4 = t$, the fundamental form g_{ik} of the general electrostatic solution [18] of Einstein's

unified field theory in the Hermitian version reads

$$g_{ik} = \begin{pmatrix} -1 & 0 & 0 & a \\ 0 & -1 & 0 & b \\ 0 & 0 & -1 & c \\ -a & -b & -c & d \end{pmatrix}, \tag{3.1}$$

with

$$d = 1 + a^2 + b^2 + c^2 \tag{3.2}$$

and

$$a = i\chi_{,x},\ b = i\chi_{,y},\ c = i\chi_{,z},\quad \chi_{,xx} + \chi_{,yy} + \chi_{,zz} = 0. \tag{3.3}$$

This particular solution of Einstein's unified field theory belongs to a class of solutions depending on three coordinates [16], outlined in Appendix A.

In the "Bildraum" x, y, z, t the imaginary part $g_{[ik]}$ of this solution just looks like the general electrostatic solution of Maxwell's theory, because the "potential" χ must obey the Laplace equation. This is not, however, just a sort of "Bildraum" deception, for both equations $\mathbf{g}^{[is]}_{,s} = 0$ and $g_{[[ik],l]} = 0$ happen to be satisfied.

If we allow for singularities at the right-hand side of the field equations, the electrostatic field due to n point charges h_q, located at $x = x_q$, $y = y_q$, $z = z_q$, can be built by taking

$$\chi = -\sum_{q=1}^{n} \frac{h_q}{p_q}, \tag{3.4}$$

where

$$p_q = [(x - x_q)^2 + (y - y_q)^2 + (z - z_q)^2]^{1/2}. \tag{3.5}$$

We expect that, if we use singularities to represent charges and currents, Einstein's Hermitian extension of the theory of general relativity should give more information than Maxwell's equations do: it should predict also the equations of motion of charges and currents, i.e., in the case of the general electrostatic solution, the law for the electrostatic equilibrium of the charges. We have agreed to represent charges by singularities, and previous experience with the problem of motion in general relativity has shown that the behaviour of the field in an infinitesimal neighbourhood of the singularities that represent the masses needs to be restricted in order to get the equations of motion. In their ground-breaking paper of 1949, Einstein and Infeld [19] did show that the equations of motion of n massive particles could be obtained by approximation methods from the vacuum field equations of general relativity alone if the metric was required to be spherically symmetric in the infinitesimal neighbourhood of each of the n particles.

We aim at imposing the same condition to the exact electrostatic solution for which, according to (3.4), χ looks in the "Bildraum" like the field of n point charges, but of course imposing the spherical symmetry can only be

done if we know what symmetric tensor represents the metric in our theory. We assume at first that the metric be given by $g_{(ik)}$.

Let Greek indices label henceforth the coordinates $x^1 = x$, $x^2 = y$, $x^3 = z$; then, according to (3.1), $g_{(\mu\nu)}$ acts as spatial metric. It is just the Euclidean one, hence it is always spherically symmetric in the infinitesimal neighbourhood of each charge, whatever the mutual positions of the n pointlike charges h_q may be. This is not the case, however, for the only nonvanishing component of $g_{(ik)}$ left, i.e. $g_{44} = d$. With our coordinates it reads:

$$d = 1 - \sum_{q=1}^{n} \frac{h_q^2}{p_q^4} \tag{3.6}$$

$$- \sum_{q,q'=1}^{n(q\neq q')} h_q h_{q'} \frac{(x-x_q)(x-x_{q'}) + (y-y_q)(y-y_{q'}) + (z-z_q)(z-z_{q'})}{p_q^3 p_{q'}^3}.$$

Let us examine d in the infinitesimal neighbourhood of, say, the qth charge. Due to the cross terms in the second line of (3.6), in general d will not tend to a spherically symmetric behaviour when the qth charge is approached; it will do so only provided that the other charges are of such strengths $h_{q'}$ and at such spatial positions $x_{q'}$, $y_{q'}$, $z_{q'}$ that:

$$\sum_{q'\neq q}^{n} h_{q'} \frac{x_q - x_{q'}}{r_{qq'}^3} = \sum_{q'\neq q}^{n} h_{q'} \frac{y_q - y_{q'}}{r_{qq'}^3} = \sum_{q'\neq q}^{n} h_{q'} \frac{z_q - z_{q'}}{r_{qq'}^3} = 0, \tag{3.7}$$

where $r_{qq'}$ is the Euclidean distance between the two charges q and q', the one measured by the components $g_{(\mu\nu)}$ of the metric $g_{(ik)}$ chosen above. Equations (3.1) and (3.7) assert that, if $g_{(ik)}$ is the metric tensor of Einstein's unified field theory, the equilibrium conditions for n point electric charges at rest predicted by an exact solution of that theory are just the same as the ones occurring in the electrostatics of Coulomb.

4. Choosing the metric of Einstein's unified field theory.

In the previous Section we have tentatively chosen $g_{(ik)}$ to be the metric of the theory in order to obtain a preliminary reading of its possible content, but we have provided no theoretical argument for this choice. The very fact that, if $g_{(ik)}$ is taken as metric, then Einstein's unified field theory contains an exact replica of Coulomb's electrostatics is not an argument of a general character, and may well be misleading. Our choice must stand on general theoretical arguments [20], dealing with the dynamics of waves and particles predicted by the theory. One such argument was provided for Einstein's unified field theory by Lichnerowicz [11, 12]. As a consequence of his study of the Cauchy problem in Einstein's unified field theory, he concluded that the metric l^{ik} appearing in the eikonal equation

$$l^{ik}\partial_i f \partial_k f = 0 \tag{4.1}$$

for the wave surfaces of the theory had to be

$$l^{ik} = g^{(ik)}, \tag{4.2}$$

or, one must add, any metric conformally related to $g^{(ik)}$. Therefore, the argument by Lichnerowicz excludes the choice of $g_{(ik)}$ as the metric of the theory.

If we adhere to Einstein's and Schrödinger's original idea that, since the theory did represent the completion of the theory of 1915, there was no need to append sources at the right-hand side of equations (2.2)-(2.5), no further argument seems available for further restricting the choice of the metric because, during the decades elapsed since the theory was first proposed, no tenable identification of physical entities in exact, everywhere regular solutions was achieved, hence no determination of the motion of matter predicted by the theory can be obtained from the contracted Bianchi identities.

This is no longer true if, in keeping with what appears from the exact solutions, one accepts the idea of appending sources to the field equations of Einstein's unified field theory, just like one does in general relativity, as first proposed in the seminal work of Hély [13, 14], outlined in Appendix B. But the analysis of the exact solutions [18] shows that there is merit if, by extending the approach of Hély, while retaining his choice of the metric, sources are appended to all the original field equations (2.2)-(2.5). The way for doing so has already been found [21]. In order to preserve the Hermitian symmetry of the equations also when the four-current density $\mathbf{j}^i = \frac{1}{4\pi}\mathbf{g}^{[is]}_{,s}$ is nonvanishing, it is necessary to substitute the symmetrised Ricci tensor of Borchsenius [22] for the plain Ricci tensor (2.6). However, since the symmetrised Ricci tensor reduces to the plain one wherever $\mathbf{j}^i = 0$, the way for adding sources of [21] allows retaining the original equations (2.2)-(2.5) in vacuo. It is reported in Appendix C; it entails that the metric must be the one chosen by Hély, defined in Appendix B by the equation

$$s^{ik} = \frac{\sqrt{-g}}{\sqrt{-s}} g^{(ik)}, \tag{4.3}$$

where s^{ik} is the inverse of Hély's metric s_{ik}, and $s \equiv \det(s_{ik})$. The dynamical equations (C.22) for charged matter obtained in Appendix C from the contracted Bianchi identities thus happen to require as metric just one among the conformally related metrics that, according to Lichnerowicz [11, 12], must enter the eikonal equation (4.1). Therefore, if Hély's metric s_{ik} is adopted in Einstein's unified field theory with sources, the dynamics of both waves and particles is ruled by one and the same metric. This was not the case with the former, tentative choice of $g_{(ik)}$ as metric tensor.

5. The general electrostatic solution when the metric is s_{ik}.

Let us reassess the general electrostatic solution (3.1)-(3.3) under the assumption that the metric is the tensor s_{ik} defined by (4.3). Due to (3.3),

for that solution s_{ik} reads

$$s_{ik} = \sqrt{d}\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - \frac{1}{\sqrt{d}}\begin{pmatrix} \chi_{,x}\chi_{,x} & \chi_{,x}\chi_{,y} & \chi_{,x}\chi_{,z} & 0 \\ \chi_{,x}\chi_{,y} & \chi_{,y}\chi_{,y} & \chi_{,y}\chi_{,z} & 0 \\ \chi_{,x}\chi_{,z} & \chi_{,y}\chi_{,z} & \chi_{,z}\chi_{,z} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{5.1}$$

hence the square of the line element can be written as

$$\mathrm{d}s^2 = s_{ik}\mathrm{d}x^i\mathrm{d}x^k = -\sqrt{d}\left(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 - \mathrm{d}t^2\right) - \frac{1}{\sqrt{d}}(\mathrm{d}\chi)^2. \tag{5.2}$$

If only one point charge h is present in the "Bildraum" x, y, z, say, at the origin of the coordinates, the "potential" χ is

$$\chi = -\frac{h}{(x^2+y^2+z^2)^{1/2}}. \tag{5.3}$$

In this case s_{ik} is spherically symmetric, like $g_{(ik)}$, but an essential difference in behaviour appears. In fact, the manifold on which $g_{(ik)}$ is defined extends to the full representative space x, y, z, although $g_{(ik)}$ becomes a negative definite metric inside the spherical surface where $d = 0$. Let us remark in passing that the quantity d appears as the g_{44} component of the fundamental tensor (3.1), but has an invariant character, since it can be written as $d = 1 - \frac{1}{2}g^{[ik]}g_{[ik]}$. When the metric is given by s_{ik}, the surface $d = 0$ instead constitutes the inner border of the manifold, since the square root of d occurs in (5.2). With this metric, the charge can no longer reside at the origin of the coordinates; it must stay on the sphere $d = 0$, whose squared coordinate radius is $r_h^2 = x^2 + y^2 + z^2 = |h|$. But is the surface $d = 0$ of the representative space a surface also in the metric sense? Let us consider the spheres centered at the origin, whose coordinate radius is larger than r_h. They are equipotential surfaces; we have in fact

$$\mathrm{d}\chi = 0 \tag{5.4}$$

for any infinitesimal, spatial displacement $\mathrm{d}x^\mu$ constrained to occur on one of these spheres. Therefore the last term of (5.2) is always zero on each sphere, and the residual square of the spatial interval tends to a vanishing value when measured on a sphere whose coordinate radius approaches r_h, since it contains the factor $\sqrt{d}$. In this sense, one can conclude that the surface $d = 0$ is in fact a point, hence the charge associated with the "potential" (5.3) can be deemed pointlike also when its spatial dimension is measured by the metric s_{ik}.

If more than one point charge is present in the "Bildraum", and the "potential" χ is no longer given by (5.3), but by (3.4) and (3.5), we may have n closed surfaces on which $d = 0$ but, whatever the positions of the

point charges in the representative space may be, they will not be equipotential surfaces. Therefore, since $\mathrm{d}\chi \neq 0$ when $d = 0$, the last term of the squared interval (5.2) will be infinite, hence there will be no room for point charges and for their equilibrium conditions in the metric sense. At variance with what occurs when the metric is assumed to be $g_{(ik)}$, introducing point charges in the "Bildraum" is not the right way for getting them eventually.

The very form of (5.2) suggests however an alternative choice: if χ is just the potential that the electrostatics of Coulomb attributes to n charged conductors, and their closed surfaces are so chosen that $d > 0$ on each of them, the last term of (5.2) will always be zero for displacements occurring on these surfaces. We can then imagine altering the shape and the relative positions of these charged surfaces in the representative space by always retaining their equipotential quality, just as it would spontaneously occur in an actual experiment done with charged conductors. If we succeed in this way in getting $d = 0$ on each of these surfaces, according to (5.2) the charges will be pointlike in the metric sense, and they will be in their mutual positions of equilibrium, since the metric s_{ik} will be spherically symmetric in an infinitesimal neighbourhood of each charged point.

Solving such a mathematical problem is obviously *extra vires* when the shapes and the mutual positions of the charged surfaces in the representative space x, y, z are allowed for full generality. However, the problem becomes much simpler if, in the "Bildraum", the charged conducting surfaces are spherical and mutually far apart. In this case χ will behave just like the potential of n charged spherical conductors whose mutual separations are very large with respect to the radii of the spheres, and will closely approximate, outside the spheres, the potential (3.4) of n point charges h_q residing at the center of the spheres. If the centers are not in the positions of equilibrium given by (3.7), the mutual inductions that render the conducting spheres equipotential, thereby ensuring the vanishing of the last term of the interval (5.2) on each charged surface, will produce an inhomogeneous surface charge density. Therefore, due to Coulomb's theorem, the value of d on each surface will not be constant.

If instead the centers are in the positions given by (3.7), the mutual inductions will become very small since, in the representative space, the radii of the spheres have been chosen to be very small with respect to their mutual separations. Therefore the value of d will be nearly constant on each charged surface, i.e. the interval will be nearly spherically symmetric in an infinitesimal neighbourhood of the surface. By taking spheres with smaller and smaller radii we can thus approach both the condition of spherical symmetry and the condition $d = 0$, that ensures the pointlike character of the charges.

Of course we cannot attain an exact result in the sense of a limit, because the radii of the spheres must remain finite in the representative space, to avoid that d become negative. However, since the largest electric fields observed until now do not seem to affect the metric properties of space

in a considerable way, we are ensured that the approximate conditions of equilibrium and the approximate pointlike structure of the charges obtained by the procedure outlined above are precise enough when confronted with the most stringent empirical constraints.

We eventually remark that the condition that $\mathrm{d}\chi$ be vanishing on the surfaces where the charges reside is equivalent to the invariant condition that the metric (5.1) be conformally flat just on these surfaces.

6. Conclusion

When sources are allowed for at the right-hand sides of the field equations (2.2)-(2.5) of Einstein's Hermitian theory of relativity in the way shown in Appendix C, the conservation identities take the physically expressive form (C.22) provided that s_{ik}, defined by (4.3), is chosen as metric.

The paradigmatic exact solution (3.1)-(3.3) happens to fully deserve the name of electrostatic solution with the metric s_{ik} too. The approximate pointlike structure of n charges and their approximate equilibrium conditions stem in fact from the purely geometrical condition of spherical symmetry in the neighbourhood of each charge also when the new, physically correct metric is adopted.

Appendix A. Solutions depending on three coordinates

Let the real symmetric tensor h_{ik} be the metric for a solution to the field equations of general relativity, which depends on the first three co-ordinates x^λ, not necessarily all spatial in character, and for which $h_{\lambda 4} = 0$. We assume Greek indices to run from 1 to 3, while Latin indices run from 1 to 4. We consider also an antisymmetric purely imaginary tensor a_{ik}, which depends on the first three co-ordinates, and we assume that its only nonvanishing components are $a_{\mu 4} = -a_{4\mu}$. Then we form the mixed tensor

$$\alpha_i{}^k = a_{il} h^{lk} = -\alpha^k_i, \tag{A.1}$$

where h^{ik} is the inverse of h_{ik}, and we define the Hermitian fundamental form g_{ik} as follows:

$$\begin{aligned} g_{\lambda\mu} &= h_{\lambda\mu}, \\ g_{4\mu} &= \alpha_4{}^\nu h_{\nu\mu}, \\ g_{44} &= h_{44} - \alpha_4{}^\mu \alpha_4{}^\nu h_{\mu\nu}. \end{aligned} \tag{A.2}$$

When the three additional conditions

$$\alpha^4{}_{\mu,\lambda} - \alpha^4{}_{\lambda,\mu} = 0 \tag{A.3}$$

are fulfilled, the affine connection Γ^i_{kl} which solves eqs. (2.2) has the nonvanishing components

$$\begin{aligned}
\Gamma^\lambda_{(\mu\nu)} &= \left\{ {\lambda \atop \mu\ \nu} \right\}_h , \\
\Gamma^\lambda_{[4\nu]} &= \alpha_4{}^\lambda{}_{,\nu} - \left\{ {4 \atop 4\ \nu} \right\}_h \alpha_4{}^\lambda + \left\{ {\lambda \atop \rho\ \nu} \right\}_h \alpha_4{}^\varrho , \\
\Gamma^4_{(4\nu)} &= \left\{ {4 \atop 4\ \nu} \right\}_h , \\
\Gamma^\lambda_{44} &= \left\{ {\lambda \atop 4\ 4} \right\}_h - \alpha_4{}^\nu \left(\Gamma^\lambda_{[4\nu]} - \alpha_4{}^\lambda \Gamma^4_{(4\nu)} \right) ;
\end{aligned} \tag{A.4}$$

we indicate with $\left\{ {i \atop k\ l} \right\}_h$ the Christoffel connection built with h_{ik}. We form now the Ricci tensor (2.6). When eqs. (2.3), i.e., in our case, the single equation

$$(\sqrt{-h}\ \alpha_4{}^\lambda h^{44})_{,\lambda} = 0, \tag{A.5}$$

and the additional conditions, expressed by eqs. (A.3), are satisfied, the components of $R_{ik}(\Gamma)$ can be written as

$$\begin{aligned}
R_{\lambda\mu} &= H_{\lambda\mu}, \\
R_{4\mu} &= \alpha_4{}^\nu H_{\nu\mu} + \left(\alpha_4{}^\nu \left\{ {4 \atop 4\ \nu} \right\}_h \right)_{,\mu} , \\
R_{44} &= H_{44} - \alpha_4{}^\mu \alpha_4{}^\nu H_{\mu\nu},
\end{aligned} \tag{A.6}$$

where H_{ik} is the Ricci tensor built with $\left\{ {i \atop k\ l} \right\}_h$. H_{ik} is zero when h_{ik} is a solution of the field equations of general relativity, as supposed; therefore, when eqs. (A.3) and (A.5) hold, the Ricci tensor, defined by eqs. (A.6), satisfies eqs. (2.4) and (2.5) of the Hermitian theory of relativity.

The task of solving equations (2.2)-(2.5) reduces, under the circumstances considered here, to the simpler task of solving eqs. (A.3) and (A.5) for a given h_{ik}.

We eventually note that the method applies also to Schrödinger's purely affine theory [7].

Appendix B. Hély's proposal for the metric and for the sources

When equations (2.2) and (2.3) of Einstein's unified field theory hold, the contracted Bianchi identities take the form [6]

$$\left[\sqrt{-g} \left(g^{ik} R_{il} + g^{ki} R_{li} \right) \right]_{,k} = \sqrt{-g} g^{ik} R_{ik,l}, \tag{B.1}$$

or else

$$\begin{aligned}
&\left(2\sqrt{-g} g^{(ik)} R_{(il)} \right)_{,k} - \sqrt{-g} g^{(ik)} R_{(ik),l} \\
&= \sqrt{-g} g^{[ik]} \left(R_{[ik],l} + R_{[kl],i} + R_{[li],k} \right) .
\end{aligned} \tag{B.2}$$

In his paper [13], entitled *"Sur la représentation d'Einstein du champ unitaire"*, Hély introduces a symmetric tensor s^{ik} such that

$$\sqrt{-s}s^{ik} = \sqrt{-g}g^{(ik)}, \tag{B.3}$$

where s is the determinant of the tensor s_{ik}, and $s^{ik}s_{il} = \delta^k_l$. By introducing this tensor in the left-hand side of (B.2) something unexpected occurs. Hély finds in fact[1] that

$$\begin{aligned} &\left(2\sqrt{-g}g^{(ik)}R_{(il)}\right)_{,k} - \sqrt{-g}g^{(ik)}R_{(ik),l} \\ &= \left(2\sqrt{-s}s^{ik}R_{(il)}\right)_{;k} - \left(\sqrt{-s}s^{ik}R_{(ik)}\right)_{;l}, \end{aligned} \tag{B.4}$$

where the semicolon stands for the covariant differentiation with respect to the Christoffel symbols built with s_{ik}. Hence the contracted Bianchi identities of Einstein's nonRiemannian extension of the vacuum general relativity of 1915 are shown by Hély to admit a sort of Riemannian rewriting in terms of the metric s_{ik}:

$$\begin{aligned} &\left(s^{ik}R_{(il)} - \frac{1}{2}\delta^k_l s^{pq}R_{(pq)}\right)_{;k} \\ &= \frac{1}{2}\sqrt{\frac{g}{s}}g^{[ik]}\left(R_{[ik],l} + R_{[kl],i} + R_{[li],k}\right), \end{aligned} \tag{B.5}$$

provided that equations (2.2) and (2.3) are satisfied. However, since in Einstein's theory equations (2.4) and (2.5) need to be satisfied too, the contracted identities in the form (B.5) appear devoid of physical sense, because both sides happen to be vanishing.

It has been already reminded that, according both to Einstein and to Schrödinger, the equations (2.2)-(2.5) did represent the completion of the equations of 1915, hence no phenomenological sources should be admitted at their right-hand sides. The form of (B.5) was however so suggestive for Hély, that he dared challenging the opinion mentioned above, and published a sequel [14], entitled *"Sur une généralisation immédiate des équations d'Einstein"*, to the previously recalled paper [13], in which phenomenological sources are appended to the right-hand sides of both (2.4) and (2.5). Thereby a nontrivial content is given to (B.5) as conservation identity, and a tentative physical interpretation to the whole theory is advanced.

In [14] Hély proposes substituting the definitions:

$$R_{(ik)} = T_{ik}, \tag{B.6}$$

$$R_{[ik],l} + R_{[kl],i} + R_{[li],k} = 4\pi J_{ikl}, \tag{B.7}$$

where T_{ik} is a symmetric tensor, while J_{ikl} is a totally antisymmetric one, for the equations (2.4) and (2.5). By availing of s^{ik} for raising indices, (B.5)

[1] In Hély's papers, f_{ik} stands for our s_{ik}, and the real nonsymmetric version of Einstein's unified field theory is considered. His equations, however, remain formally unaffected when the complex Hermitian version of the theory is considered instead.

comes to read:

$$\left(T_l^k - \frac{1}{2}\delta_l^k T_s^s\right)_{;k} = 2\pi\sqrt{\frac{g}{s}}g^{[ik]}J_{ikl}. \tag{B.8}$$

Then the right-hand side of (B.8) describes the Lorentz force exerted by the antisymmetric field $\sqrt{g/s}\ g^{[ik]}$ on the conserved current J_{ikl}. Pending the final verdict coming from the solutions to the field equations, this electromagnetic interpretation appears to be a consistent one: by looking at equations (B.7) and (2.3) together, Hély is entitled to assert that, in the generalization of Einstein's theory proposed by him, $\sqrt{g/s}\ g^{[ik]}$ and $R_{[ik]}$ are proportional to the duals of the electromagnetic fields $(\overrightarrow{E}, \overrightarrow{B})$ and $(\overrightarrow{D}, \overrightarrow{H})$ respectively. Like in the theory of 1915, the Lorentz force appears to be due to the nonconservation of some stress-energy-momentum tensor associated with the fields, whose expression is however much more complicated that in the Maxwellian case, because the constitutive equation linking $\sqrt{g/s}\ g^{[ik]}$ and $R_{[ik]}$ is a very complicated, differential relation, without counterpart in classical electromagnetism.

Appendix C. Generalization of Hély's approach

The way of appending sources given in [21] will be recalled here in full detail, in order to allow for the comparison with Hély's original proposal. On a four-dimensional, real manifold, let $\mathbf{g}^{ik}$ be a Hermitian contravariant tensor density

$$\mathbf{g}^{ik} = \mathbf{g}^{(ik)} + \mathbf{g}^{[ik]}. \tag{C.1}$$

We also endow the manifold with a general, complex affine connection

$$W_{kl}^i = W_{(kl)}^i + W_{[kl]}^i; \tag{C.2}$$

for the Riemann curvature tensor built with this connection:

$$R^i{}_{klm}(W) = W_{kl,m}^i - W_{km,l}^i - W_{al}^i W_{km}^a + W_{am}^i W_{kl}^a, \tag{C.3}$$

two distinct nonvanishing contractions [23], namely $R_{ik}(W) = R^p{}_{ikp}(W)$ and $A_{ik}(W) = R^p{}_{pik}(W)$ do exist. But also the transposed affine connection $\tilde{W}_{kl}^i = W_{lk}^i$ shall be taken into account: from it, the Riemann curvature tensor $R^i{}_{klm}(\tilde{W})$ and its two contractions $R_{ik}(\tilde{W})$ and $A_{ik}(\tilde{W})$ can be formed as well. We aim at following the pattern of general relativity, in which the Lagrangian density $\mathbf{g}^{ik}R_{ik}$ is considered, but now any linear combination $\bar{R}_{ik}$ of the four above-mentioned contractions can be envisaged. A good choice [22], for reasons that will become apparent much later, is

$$\bar{R}_{ik}(W) = R_{ik}(W) + \frac{1}{2}A_{ik}(\tilde{W}). \tag{C.4}$$

Let us try to endow the theory with sources in the form of a nonsymmetric tensor P_{ik} and of a current density $\mathbf{j}^i$, coupled to $\mathbf{g}^{ik}$ and to the vector

$W_i = W^l_{[il]}$ respectively. The Lagrangian density

$$\mathbf{L} = \mathbf{g}^{ik}\bar{R}_{ik}(W) - 8\pi\mathbf{g}^{ik}P_{ik} + \frac{8\pi}{3}W_i\mathbf{j}^i \tag{C.5}$$

is thus arrived at. By performing independent variations of the action $\int \mathbf{L}d\Omega$ with respect to W^p_{qr} and to $\mathbf{g}^{ik}$ with suitable boundary conditions we obtain the field equations

$$\begin{aligned} -\mathbf{g}^{qr}_{,p} + \delta^r_p\mathbf{g}^{(sq)}_{,s} - \mathbf{g}^{sr}W^q_{sp} - \mathbf{g}^{qs}W^r_{ps} + \delta^r_p\mathbf{g}^{st}W^q_{st} + \mathbf{g}^{qr}W^t_{pt} \\ = \frac{4\pi}{3}(\mathbf{j}^r\delta^q_p - \mathbf{j}^q\delta^r_p), \end{aligned} \tag{C.6}$$

and

$$\bar{R}_{ik}(W) = 8\pi P_{ik}. \tag{C.7}$$

By contracting eq. (C.6) with respect to q and p one obtains

$$\mathbf{g}^{[is]}_{,s} = 4\pi\mathbf{j}^i, \tag{C.8}$$

a promising outcome, but a problem too. In fact, the very existence of (C.8) tells that, for given $\mathbf{j}^i$, equation (C.6) cannot determine the affine connection W^i_{kl} uniquely in terms of $\mathbf{g}^{ik}$: (C.6) is in fact invariant under the projective transformation $W'^i_{kl} = W^i_{kl} + \delta^i_k\lambda_l$, where λ_l is an arbitrary vector field. Moreover eq. (C.7) is invariant under the transformation

$$W'^i_{kl} = W^i_{kl} + \delta^i_k\mu_{,l}, \tag{C.9}$$

where μ is an arbitrary scalar. Both equation (C.8) and the invariance under (C.9) are reminiscent of electromagnetism as we know it. We can write

$$W^i_{kl} = \Gamma^i_{kl} - \frac{2}{3}\delta^i_k W_l, \tag{C.10}$$

where Γ^i_{kl} is another affine connection, by definition constrained to yield $\Gamma^l_{[il]} = 0$. Then eq. (C.6) becomes

$$\mathbf{g}^{qr}_{,p} + \mathbf{g}^{sr}\Gamma^q_{sp} + \mathbf{g}^{qs}\Gamma^r_{ps} - \mathbf{g}^{qr}\Gamma^t_{(pt)} = \frac{4\pi}{3}(\mathbf{j}^q\delta^r_p - \mathbf{j}^r\delta^q_p), \tag{C.11}$$

and allows us to determine Γ^i_{kl} uniquely, under very general conditions, in terms of $\mathbf{g}^{ik}$. When eq. (C.10) is substituted in eq. (C.7), the even and the alternating part of the latter come to read:

$$\bar{R}_{(ik)}(\Gamma) = 8\pi P_{(ik)} \tag{C.12}$$

$$\bar{R}_{[ik]}(\Gamma) = 8\pi P_{[ik]} - \frac{1}{3}(W_{i,k} - W_{k,i}) \tag{C.13}$$

respectively. An unsurmountable difficulty appears in equation (C.13). In fact [23], wherever a source term is nonvanishing, a field equation loses its meaning, and reduces to a definition of some property of matter in terms of geometrical entities; it is quite obvious that such a definition must be unique. This necessary occurrence happens with eqs. (C.11), (C.8) and (C.12), but it does not happen with eq. (C.13). This equation only prescribes that

$\bar{R}_{[ik]}(\Gamma) - 8\pi P_{[ik]}$ is the curl of an arbitrary vector $W_i/3$; it is just equivalent to the four equations

$$\bar{R}_{[[ik],l]}(\Gamma) = 8\pi P_{[[ik],l]}, \tag{C.14}$$

hence it cannot specify $P_{[ik]}$ uniquely. Therefore we must dismiss the redundant tensor $P_{[ik]}$, and assume henceforth that matter is defined by the symmetric tensor $P_{(ik)}$, by the current density $\mathbf{j}^i$ and by the current

$$K_{ikl} = \frac{1}{8\pi}\bar{R}_{[[ik],l]}; \tag{C.15}$$

both $\mathbf{j}^i$ and K_{ikl} are conserved quantities by definition. The analogy with the general relativity of 1915, to which the present theory formally reduces when $\mathbf{g}^{[ik]} = 0$, suggests rewriting eq. (C.12) as

$$\bar{R}_{(ik)}(\Gamma) = 8\pi(T_{ik} - \frac{1}{2}s_{ik}s^{pq}T_{pq}), \tag{C.16}$$

where $s_{ik} = s_{ki}$ is the still unchosen metric tensor of the theory, $s^{il}s_{kl} = \delta^i_k$, and the symmetric tensor T_{ik} will be tentatively assumed to play the rôle of energy tensor.

Equations (C.11), (C.8), (C.16) and (C.15) reduce to the equations (2.2)-(2.5) of Einstein's unified field theory when sources are absent, since then $\bar{R}_{ik}(\Gamma)$=$R_{ik}(\Gamma)$; moreover they enjoy the property of transposition invariance even when sources are present. If $\mathbf{g}^{ik}$ is Hermitian, like it was assumed, Γ^i_{kl}, as defined by equation (C.11), is Hermitian too, and the same property is enjoyed also by $\bar{R}_{ik}(\Gamma)$. Let these quantities represent a solution with the sources T_{ik}, $\mathbf{j}^i$ and K_{ikl}. The transposed quantities $\tilde{\mathbf{g}}^{ik} = \mathbf{g}^{ki}$, $\tilde{\Gamma}^i_{kl} = \Gamma^i_{lk}$ and $\bar{R}_{ik}(\tilde{\Gamma})$= $\bar{R}_{ki}(\Gamma)$ then provide another solution, endowed with the sources $\tilde{T}_{ik} = T_{ik}, \tilde{\mathbf{j}}^i = -\mathbf{j}^i$ and $\tilde{K}_{ikl} = -K_{ikl}$. Such a desirable property is a consequence of the choice made for $\bar{R}_{ik}$. These equations suggest interpreting Einstein's unified field theory with sources as a gravoelectrodynamics in a polarizable continuum, allowing for both electric and magnetic currents. The study of the conservation identities confirms the idea and leads at the same time to the determination of the metric tensor s_{ik} that appears in equation (C.16). One considers the invariant integral

$$I = \int \left[\mathbf{g}^{ik}\bar{R}_{ik}(W) + \frac{8\pi}{3}W_i\mathbf{j}^i\right] d\Omega. \tag{C.17}$$

From it, when eq. (C.6) is assumed to hold, by means of an infinitesimal coordinate transformation the four identities

$$\begin{aligned} -(\mathbf{g}^{is}\bar{R}_{ik}(W) + \mathbf{g}^{si}\bar{R}_{ki}(W))_{,s} + \mathbf{g}^{pq}\bar{R}_{pq,k}(W) \\ +\frac{8\pi}{3}\mathbf{j}^i(W_{i,k} - W_{k,i}) = 0 \end{aligned} \tag{C.18}$$

are obtained. This equation can be rewritten as

$$-2(\mathbf{g}^{(is)}\bar{R}_{(ik)}(\Gamma))_{,s}+\mathbf{g}^{(pq)}\bar{R}_{(pq),k}(\Gamma) = 2\mathbf{g}^{[is]}_{\ ,s}\bar{R}_{[ik]}(\Gamma)+\mathbf{g}^{[is]}\bar{R}_{[[ik],s]}(\Gamma), \tag{C.19}$$

where the redundant variable W^i_{kl} no longer appears. Let us remind of eq. (C.16), and assume, like Hély did [13, 14], that the metric tensor is defined by the equation

$$\sqrt{-s}s^{ik}=\mathbf{g}^{(ik)}, \tag{C.20}$$

where $s=\det(s_{ik})$; we shall use henceforth s^{ik} and s_{ik} to raise and lower indices, $\sqrt{-s}$ to produce tensor densities out of tensors. We define then

$$\mathbf{T}^{ik}=\sqrt{-s}s^{ip}s^{kq}T_{pq}, \tag{C.21}$$

and the weak identities (C.19), when all the field equations hold, will take the form

$$\mathbf{T}^{ls}_{;s}=\frac{1}{2}s^{lk}(\mathbf{j}^i\bar{R}_{[ki]}(\Gamma)+K_{iks}\mathbf{g}^{[si]}), \tag{C.22}$$

where the semicolon indicates the covariant derivative done with respect to the Christoffel connection

$$\left\{{i \atop k\ l}\right\}=\frac{1}{2}s^{im}(s_{mk,l}+s_{ml,k}-s_{kl,m}) \tag{C.23}$$

built with s_{ik}. Our earlier impression is confirmed by eq. (C.22): the theory, built in terms of a non-Riemannian geometry, entails a gravoelectrodynamics in a dynamically polarized Riemannian spacetime, for which s_{ik} is the metric. Like in Hély's proposal [14] of 1954, the relationship between electromagnetic inductions and fields is governed by the field equations in a quite novel and subtle way, with respect to the one prevailing, say, in the electromagnetic vacuum of the so-called Einstein-Maxwell theory; with aftersight, one may well assert that this novelty has constituted, besides the choice of the metric, another major stumbling block in the understanding of the theory.

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Dipartimento di Fisica "A. Volta" and CNR, Pavia, Italy
E-mail address: Antoci@fisicavolta.unipv.it

Astrophysikalisches Institut Potsdam, Potsdam, Germany
E-mail address: deliebscher@aip.de

Dipartimento di Fisica "A. Volta", Pavia, Italy
E-mail address: Mihich@fisicavolta.unipv.it