# THE ISSUE OF PHOTONS IN DIELECTRICS: HAMILTONIAN VIEWPOINT

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ABSTRACT. The definition of the photon in the vacuum of general relativity provided by Kermack et al. and by Synge is extended to nondispersive, nonhomogeneous, isotropic dielectrics in arbitrary motion by Hamiltonian methods that rely on Gordon's effective metric. By these methods the old dilemma, whether the momentum-energy vector of the photon in dielectrics is timelike or spacelike in character, is shown to reappear under a novel guise.

## 1. INTRODUCTION

Despite a widespread conviction, the concept of photon in vacuo does not pertain exclusively to quantum physics. Classical general relativity has its say on the subject, as it was beautifully shown long ago by Kermack et al. [1] and by Synge [2]. By availing of Hamiltonian methods, and of the reduction to the vacuum case of the geometrical optics of dielectrics operated by Gordon [3] and by Pham Mau Quan [4], the problem of the definition of the photon in dielectrics is considered anew in the present paper.

The abstract formulation [5] of Hamilton's theory of rays and waves is first recalled (Section 2). The metric is then introduced, and the particular Hamiltonians that apply to the case of the vacuum (Section 3), and of isotropic, nondispersive dielectrics (Section 4) are considered. Synge's derivation of the proportionality between energy and frequency for the photon in the vacuum of general relativity is then displayed (Section 5). His argument, when applied to the case of dielectrics, leads to a twofold possibility: one must choose either a timelike photon (Section 6), for which however the proportionality between energy and frequency breaks down in nonhomogeneous dielectrics, or a spacelike photon (Section 7) for which the proportionality is always ensured. The latter option however requires, at variance with what occurs in mechanics, that an entity endowed with a spacelike momentum-energy vector be associated to a timelike ray. The existence of this twofold possibility is akin to the ancient, unsolved dilemma in the electromagnetism of continua, whether the energy tensor of the electromagnetic field in dielectrics should be given either the expression proposed by Abraham [6] or the one formulated by Minkowski [7].

## 2. Hamilton's theory in abstract form

Like Maxwell's equations, also the equations of Hamilton's theory of rays and waves [5] lie very deep in the conceptual structure of physics. In fact, as it occurs with Maxwell's equations [8], these equations can be written without availing of either an affine or a metric connection: a bare eight-dimensional manifold suffices, equipped only with coordinates $x^i$ and $y_i$ [1]. In this manifold a 7-surface $\Sigma$ is assumed to exist, whose equation is

$$(2.1) \qquad\qquad H(x^i, y_i) = 0.$$

By following Synge's account, we consider a curve $\Gamma$ joining the points $A$ and $B$ on $\Sigma$, and the integral

$$(2.2) \qquad\qquad I = \int_A^B y_i \mathrm{d}x^i.$$

We propose ourselves to find the extremals of this integral with the varied curve always lying on $\Sigma$. To this end a parameter $u$ is introduced, whose values at $A$ and at $B$ are fixed for all the curves, and we get rid of the side condition (2.1) by considering the variation of the integral

$$(2.3) \qquad\qquad J = \int_A^B (y_i \mathrm{d}x^i - \lambda H \mathrm{d}u),$$

where $\lambda(u)$ is a Lagrange multiplier. A generic variation of $J$ reads

$$(2.4) \qquad\qquad \delta J = \left[ y_i \delta x^i \right]_B^A$$
$$+ \int_A^B \left( \delta y_i \mathrm{d}x^i - \delta x^i \mathrm{d}y_i - H \delta \lambda \mathrm{d}u - \lambda \frac{\partial H}{\partial x^i} \delta x^i \mathrm{d}u - \lambda \frac{\partial H}{\partial y_i} \delta y_i \mathrm{d}u \right).$$

It is asked that $\delta J = 0$ under arbitrary variations of $\delta x^i$, $\delta y_i$, $\delta \lambda$, provided that $\delta x^i$ vanish at $A$ and at $B$. One finds that the extremals must obey the equations

$$(2.5) \qquad\qquad \frac{\mathrm{d}x^i}{\mathrm{d}u} = \lambda \frac{\partial H}{\partial y_i}, \ \frac{\mathrm{d}y_i}{\mathrm{d}u} = -\lambda \frac{\partial H}{\partial x^i}, \ H = 0.$$

Since the variation does not determine $\lambda(u)$, one can choose the parameter $u$ in order to write the equations of the extremals in the Hamiltonian form

$$(2.6) \qquad\qquad \frac{\mathrm{d}x^i}{\mathrm{d}u} = \frac{\partial H}{\partial y_i}, \ \frac{\mathrm{d}y_i}{\mathrm{d}u} = -\frac{\partial H}{\partial x^i}, \ H = 0.$$

These equations can be given another interpretation, in terms of a four-dimensional manifold whose coordinates are $x^i$. In the new interpretation, equation (2.1) no longer refers to a 7-surface $\Sigma$, but to a set of 3-spaces, each one associated to a point $x^i$. We want that the theory be invariant under arbitrary transformations of the $x$-coordinates, hence we must think of the $y_i$ as a covariant four-vector defined on the manifold. Considered in the original eight-dimensional space, an extremal is a curve on $\Sigma$. In the

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[1]Latin indices run henceforth from 1 to 4.

four-dimensional manifold whose coordinates are $x^i$, it appears as a curve $x^i = x^i(u)$ with an associated vector field $y_i = y_i(u)$. Let us consider from the four-dimensional standpoint the integral (2.2) along an extremal joining the points $A(x'^i)$ and $B(x^i)$, and call it $f(x'^i, x^i)$. It is nothing but Hamilton's characteristic function. If the points $A$ and $B$ are varied and the integral for the new extremal is compared to the integral for the previous one, since the integral in the second row of (2.4) vanishes for the extremals, one finds

$$(2.7) \qquad\qquad \delta f = \left[ y_i \delta x^i \right]_A^B.$$

Provided that $\delta x'^i, \delta x^i$ can be chosen arbitrarily, it must be

$$(2.8) \qquad\qquad \frac{\partial f}{\partial x^i} = y_i, \ \frac{\partial f}{\partial x'^i} = -y'_i.$$

Inserting the first of (2.8) in (2.1) produces the Hamilton-Jacobi equation

$$(2.9) \qquad\qquad H\left( x^i, f_{,i} \right) = 0, \ \text{where} \ f_{,i} \equiv \frac{\partial f}{\partial x^i}.$$

If the vector field $y_i$ defined by the extremals is such that the circulation

$$(2.10) \qquad\qquad \oint_C f_{,i} dx^i = 0$$

for an arbitrary closed curve $C$ lying within some (two, three or four dimensional) domain $D$, the extremals form what Synge calls a coherent system and are named rays. For a coherent system, the integral

$$(2.11) \qquad\qquad I(A,B) = \int_A^B f_{,i} dx^i$$

between a fixed point $A$ and a variable point $B$ does not depend on the path of integration. The subspace of $D$ that is found when $B$ varies but $I(A,B)$ maintains a constant value $v$ is called a wave; by varying the parameter $v$ we get a set of waves; changing the starting point $A$ does not change the set; it merely changes its parametrisation.

## 3. Introducing the metric. Null waves and null rays.

For associating the quantities of the theory with the quantities of experience we need a metric. We assume that $g_{ik}$ is the usual pseudo-Riemannian metric of general relativity, that can be locally reduced through a suitable transformation to the Minkowski form $\eta_{ik} \equiv \text{diag}(1,1,1,-1)$. Then we can introduce the timelike world-line of an observer endowed with four-velocity $u^i$ and define the frequency of the wave with respect to that observer:

$$(3.1) \qquad\qquad \nu = f_{,i} u^i.$$

We can also consider whether $f_{,i}$ is spacelike, timelike or null. In the first case $\nu$ can vanish locally, while this is impossible in the second case. One can also show [5] that in the first case the phase velocity of the wave is inferior to the fundamental velocity introduced by the metric, while the converse is

true in the second case. In the third case the wave propagates just with the fundamental velocity. Such a null wave obeys the Hamilton-Jacobi equation

$$(3.2) \qquad H\left(x^i, f_{,i}\right) = \frac{1}{2} g^{ik} f_{,i} f_{,k} = 0.$$

Since in this case the first of Hamilton's equations (2.6) reads

$$(3.3) \qquad \frac{\mathrm{d}x^i}{\mathrm{d}u} = g^{ik} f_{,k}$$

the ray velocity $\frac{\mathrm{d}x^i}{\mathrm{d}u}$ is null:

$$(3.4) \qquad g_{ik} \frac{\mathrm{d}x^i}{\mathrm{d}u} \frac{\mathrm{d}x^k}{\mathrm{d}u} = 0.$$

The second of Hamilton's equations reads now

$$(3.5) \qquad \frac{\mathrm{d}f_{,i}}{\mathrm{d}u} = -\frac{1}{2} g^{kl}_{\phantom{kl},i} f_{,k} f_{,l}.$$

By eliminating $f_{,i}$ in (3.5) with the use of (3.3) one finds that

$$(3.6) \qquad \frac{\delta}{\delta u} \frac{\mathrm{d}x^i}{\mathrm{d}u} = 0;$$

the extremals for the Hamiltonian (3.2) are null geodesics, with the parameter $u$ of the extremal taking the role of special parameter for the geodesic.

For a coherent system one has

$$(3.7) \qquad f_{,i} \delta x^i = 0$$

when $\delta x^i$ is a displacement along the wave. Due to (3.3) one obtains

$$(3.8) \qquad g_{ik} \delta x^i \frac{\mathrm{d}x^k}{\mathrm{d}u} = 0.$$

Since, again due to (3.3)

$$(3.9) \qquad f_{,i} \frac{\mathrm{d}x^i}{\mathrm{d}u} = g^{ik} f_{,i} f_{,k} = 0$$

one finds that not only the null ray is orthogonal to the null surface, but it lies in it too. The fact that a certain curve lies in a certain surface is however an occurrence that can be ascertained without a metric; we shall remind of this property in the following, when dealing with photons in dielectrics.

## 4. The characteristics of Maxwell's equations for a nondispersive, isotropic dielectric

We define [9] the electric displacement and the magnetic field by the antisymmetric, contravariant tensor density $\mathbf{H}^{ik}$, while the electric field and the magnetic induction are given by the skew, covariant tensor $F_{ik}$. We define also the four-vectors:

$$(4.1) \qquad F_i = F_{ik} u^k, \ H_i = H_{ik} u^k,$$

where $u^i$ is the four-velocity of the medium. If the latter is isotropic in its rest frame, its linear constitutive equation reads [7]

$$(4.2) \qquad \mu H^{ik} = F^{ik} + (\epsilon\mu - 1)(u^i F^k - u^k F^i).$$

Gordon [3] noticed that (4.2) can be rewritten as

$$(4.3) \qquad \mu H^{ik} = \left[g^{ir} - (\epsilon\mu - 1)u^i u^r\right]\left[g^{ks} - (\epsilon\mu - 1)u^k u^s\right] F_{rs}.$$

Therefore he introduced the "effective metric tensor"

$$(4.4) \qquad \sigma^{ik} = g^{ik} - (\epsilon\mu - 1)u^i u^k$$

and rewrote the constitutive equation (4.2) as

$$(4.5) \qquad \mu \mathbf{H}^{ik} = \sqrt{g}\,\sigma^{ir}\sigma^{ks}F_{rs},$$

where $g \equiv -\det(g_{ik})$. Since after some simple algebra one finds [3] that

$$(4.6) \qquad \sigma = \frac{g}{\epsilon\mu},$$

where $\sigma \equiv -\det(\sigma_{ik})$, equation (4.3) can be eventually rewritten as

$$(4.7) \qquad \mathbf{H}^{ik} = \sqrt{\frac{\epsilon}{\mu}}\sqrt{\sigma}\,\sigma^{ir}\sigma^{ks}F_{rs}.$$

Therefore, apart from the factor $\sqrt{\epsilon/\mu}$, the constitutive equation in a nondispersive, isotropic dielectric medium is rendered by the effective metric $\sigma_{ik}$ exactly in the way the constitutive equation for vacuum is rendered by the pseudo-Riemannian metric $g_{ik}$. If the medium is also homogeneous in its rest frame, the factor $\sqrt{\epsilon/\mu}$ is constant, and the analogy with the vacuum case is complete. Then one can reduce to the vacuum the problem of the choice of the Lagrangian for the electromagnetic field in nondispersive, homogeneous, isotropic dielectrics. As a consequence, in such dielectrics the definition of the energy tensor of the electromagnetic field is uniquely given by Gordon's procedure [3], and in this way the general relativistic extension of Abraham's form [6] is retrieved. In his seminal paper Gordon, after finding the expression of the wave equation for tensorial fields in a pseudo-Riemannian space, dealt with the limit of geometrical optics in the footsteps of Hadamard [10] and proved that the propagation of light in a dielectric that is isotropic and homogeneous in its local rest frame must obey, besides an equation that links the amplitude $A$ and the phase $f$, the equation

$$(4.8) \qquad \sigma^{ik}f_{,i}f_{,k} = 0,$$

where $\sigma^{ik}$ is the effective metric (4.4). To this equation Gordon applied the Hamiltonian method of Section 3 to prove [2], inter alia, that "the rays in moving bodies are represented through null geodesics of the manifold with the line element $d\sigma^2 = \sigma_{ik}dx^i dx^k$".

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[2]See Ref. [3], page 455.

Gordon's proof applies to homogeneous dielectrics. In 1957, by studying the characteristics of the Maxwell-Einstein equations [4], Pham Mau Quan extended the validity of (4.8) to a nonhomogeneous medium.

## 5. PHOTONS IN THE VACUUM OF GENERAL RELATIVITY

In two remarkable papers [1, 2] Kermack et al. and Synge have shown that the concept of photon in vacuo is by no means an exclusive outcome of quantum theory, but is deeply rooted in classical general relativity. We have already availed [11] of Synge's proof of the proportionality of energy and frequency for an entity endowed with both the behaviour of a null wave and of a null particle, exchanged between two atoms in arbitrary states of motion in a gravitational field. We have rather reproducing it here, since it is necessary for the discussion of photons in dielectrics from the Hamiltonian viewpoint.

Atoms are dealt with as pointlike entities. In Figure 1 are shown the world-lines of an emitting and of an absorbing atom, joined by a null line $A_0A$. We attribute to the photon the Hamiltonian (3.2); viewed as a point particle, its equations of motion are (3.6) and (3.4).
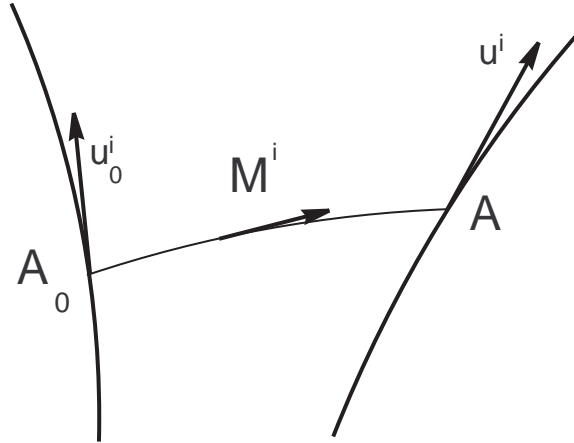


FIGURE 1. Spacetime diagram for two atoms exchanging Synge's photon.

We endow this point particle with a momentum-energy vector $M^i$ that is tangent to the photon's world-line and is parallelly propagated along it:

$$(5.1) \qquad M^i = \theta \frac{\mathrm{d}x^i}{\mathrm{d}u}, \quad \frac{\delta M^i}{\delta u} = 0,$$

where $\theta$, assumed to be positive, happens to be a constant thanks to (3.6). Since it travels with the fundamental velocity, the photon must have zero

proper mass, hence:

$$(5.2) \qquad\qquad M_i M^i = 0.$$

An atom is instead pictured as a timelike point particle endowed with the momentum-energy vector

$$(5.3) \qquad\qquad m_0 u^i = m_0 \frac{\mathrm{d}x^i}{\mathrm{d}s};$$

$m_0$ is the atom's rest mass or rest energy, while $u^i$ is its four-velocity. The change of the momentum-energy vector of the atom due to absorption of a photon is supposedly ruled by the conservation law:

$$(5.4) \qquad\qquad m_0' u'^i = m_0 u^i + M^i,$$

where $m_0'$ is the rest mass, and $u'^i$ is the four-velocity of the atom after the absorption of the photon. By rewriting the conservation law in covariant form and by multiplying the two forms term by term one finds

$$(5.5) \qquad\qquad -m'^2_0 = -m_0^2 + 2m_0 M_i u^i,$$

i.e.

$$(5.6) \qquad\qquad \frac{m'^2_0 - m_0^2}{2m_0} = -M_i u^i.$$

Since for optical processes $|m'_0 - m_0|/m_0 \ll 1$ Synge defines the energy $E$ of a photon with momentum-energy vector $M^i$ relative to an atom endowed with four-velocity $u^i$ as

$$(5.7) \qquad\qquad E = -M_i u^i.$$

This scalar quantity is numerically equal to the fourth component of $M^i$ in a locally Minkowskian coordinate system where the atom is at rest.

The special parameter $u$ along the null geodesics joining the world-lines of the absorbing and of the emitting atoms can be given a single starting value and a single terminal value on these lines, since the geodesic equation is invariant under the linear transformation $u' = au + b$, where $a$ and $b$ are constants. Under this transformation the Hamilton equations (3.3) and (3.5) are preserved too, provided that the $y_i$ undergo the corresponding transformation. The latter does not affect the structure of the waves, but only their parametrisation. Let $v$ be the parameter introduced at the end of Section 2, associated to the waves after this reparametrisation has occurred. We can write

$$(5.8) \qquad\qquad \frac{\partial}{\partial u}\left(M_i \frac{\partial x^i}{\partial v}\right) = \frac{\delta M_i}{\delta u}\frac{\partial x^i}{\partial v} + M_i \frac{\delta}{\delta u}\frac{\partial x^i}{\partial v}.$$

We have also

$$(5.9) \qquad\qquad \frac{\delta}{\delta u}\frac{\partial x^i}{\partial v} \equiv \frac{\partial^2 x^i}{\partial u \partial v} + \Gamma^i_{kl}\frac{\partial x^k}{\partial v}\frac{\partial x^l}{\partial u} = \frac{\delta}{\delta v}\frac{\partial x^i}{\partial u},$$

where $\Gamma^i_{kl}$ is the Christoffel connection built with $g_{ik}$. Since $\delta M_i/\delta u = 0$, the right hand side of (5.8) can be rewritten as

$$(5.10) \qquad M_i \frac{\delta}{\delta u} \frac{\partial x^i}{\partial v} = \theta g_{ik} \frac{\partial x^k}{\partial u} \frac{\delta}{\delta v} \frac{\partial x^i}{\partial u} = \frac{1}{2} \theta \frac{\partial}{\partial v} \left( g_{ik} \frac{\partial x^i}{\partial u} \frac{\partial x^k}{\partial u} \right) = 0.$$

Hence one finds

$$(5.11) \qquad \frac{\partial}{\partial u} \left( M_i \frac{\partial x^i}{\partial v} \right) = 0,$$

and, with reference to Figure 1:

$$(5.12) \qquad \left( M_i \frac{\partial x^i}{\partial v} \right)_{A_0} = \left( M_i \frac{\partial x^i}{\partial v} \right)_A.$$

Let $u_0^i$, $u^i$ be the four-velocities of the atoms at $A_0$ and at $A$ respectively. If $dv$ is the infinitesimal increment of the parameter $v$ when going from the null surface joining $A_0$ and $A$ to a neighbouring one, we can write:

$$(5.13) \qquad \left( \frac{\partial x^i}{\partial v} \right)_{A_0} dv = u_0^i ds_0, \quad \left( \frac{\partial x^i}{\partial v} \right)_A dv = u^i ds,$$

hence from (5.12) we get

$$(5.14) \qquad \left( M_i u^i \right)_{A_0} ds_0 = \left( M_i u^i \right)_A ds,$$

and the definition (5.7) of the energy of a photon absorbed (or emitted) by an atom allows rewriting the last equation as

$$(5.15) \qquad E_0 ds_0 = E ds.$$

From the wave model of the photon implicit in Hamilton's formulation in a coherent system one gathers that, if $ds_0$ and $ds$ are the intervals during which the emission and the absorption of the wavelike photon takes place respectively at the atoms $A_0$ and $A$, it must be:

$$(5.16) \qquad \nu_0 ds_0 = \nu ds,$$

where $\nu_0$ and $\nu$ are the scalar frequencies [3] of emission and of absorption defined by (3.1). By dividing term by term this equation and equation (5.15) we get:

$$(5.17) \qquad \frac{E_0}{\nu_0} = \frac{E}{\nu}.$$

Therefore, Hamilton's theory allows to show that if a photon is emitted by an atom and absorbed by another one in presence of a gravitational field, the ratio energy/frequency is the same for emission and for absorption. This ratio is independent of the behaviour of the gravitational field and of the state of motion of the two atoms.

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[3]The value of the scalar frequency (3.1) depends on the parametrisation, but this fact affects both sides of equation (5.16) in the same way.

## 6. Timelike photons in nondispersive, isotropic dielectrics

Thanks to Gordon's effective metric (4.4) and to the corresponding Hamiltonian (4.8) it is immediate to extend Synge's proof to nondispersive, isotropic dielectrics. The diagram in Figure 1 applies also in this case, and the $(u, v)$ parametrisation remains unaltered. The metric

$$(6.1) \qquad \sigma_{ik} = g_{ik} - \left( \frac{1}{\epsilon\mu} - 1 \right) u_i u_k$$

takes in the Hamiltonian the role that $g_{ik}$ had in the previous Section, but of course also the latter shall intervene: it is the real metric of spacetime, and *e.g.* the line elements $ds_0$ and $ds$ entering the four-velocities of the atoms and of the medium must be measured by $g_{ik}$. We define the null geodesics of the effective spacetime as

$$(6.2) \qquad \frac{\sigma\delta}{\delta u} \frac{dx^i}{du} \equiv \frac{d^2 x^i}{du^2} + \Sigma^i_{kl} \frac{dx^k}{du} \frac{dx^l}{du} = 0,$$

where

$$(6.3) \qquad \sigma_{ik} \frac{dx^i}{du} \frac{dx^k}{du} = 0,$$

and the Christoffel connection to be used is now

$$(6.4) \qquad \Sigma^i_{kl} = \frac{1}{2} \sigma^{im} \left( \sigma_{mk,l} + \sigma_{ml,k} - \sigma_{kl,m} \right).$$

Gordon's reduction to the vacuum imposes to model the photon as a null particle of the effective spacetime, and to write

$$(6.5) \qquad M^i = \theta \frac{dx^i}{du}, \quad \frac{\sigma\delta M^i}{\delta u} = 0,$$

where $\theta$, assumed to be positive, happens to be a constant thanks to (6.2). Since the photon travels with the fundamental velocity of $\sigma_{ik}$ we shall pose

$$(6.6) \qquad \sigma_{ik} M^i M^k \equiv M_{(i)} M^i = 0,$$

where we have adopted Gordon's convention of enclosing in round parentheses the indices moved with the effective metric. We can again model the interaction of the photon with the atom through the equation $m'_0 u'^i = m_0 u^i + M^i$. Since

$$(6.7) \qquad \sigma_{ik} M^i M^k = \left[ g_{ik} + \left( 1 - \frac{1}{\epsilon\mu} \right) u_i u_k \right] M^i M^k$$

$$= g_{ik} M^i M^k + \left( 1 - \frac{1}{\epsilon\mu} \right) \left( u_i M^i \right)^2 = 0,$$

one finds

$$(6.8) \qquad g_{ik} M^i M^k \equiv M_i M^i = \left( \frac{1}{\epsilon\mu} - 1 \right) \left( u_i M^i \right)^2.$$

Since $\epsilon\mu > 1$, when measured with the real spacetime metric, the momentum-energy of the photon turns out to be timelike, as it occurs to an ordinary particle, for which a rest coordinate system exists. Therefore one finds

$$(6.9) \qquad -m'^2_0 = -m_0^2 + 2m_0 M_i u^i + \left(\frac{1}{\epsilon\mu} - 1\right)\left(u_i M^i\right)^2;$$

instead of (5.5), but for optical processes the last term is negligible when compared to the remaining ones, and one can still define the energy of the photon as in equation (5.7). The argument of the previous Section can now be repeated with the due changes, by starting from the quantity

$$(6.10) \qquad \frac{\partial}{\partial u}\left(M_{(i)}\frac{\partial x^i}{\partial v}\right) = \frac{{}^\sigma\delta M_{(i)}}{\delta u}\frac{\partial x^i}{\partial v} + M_{(i)}\frac{{}^\sigma\delta}{\delta u}\frac{\partial x^i}{\partial v}.$$

One obtains

$$(6.11) \qquad \frac{\partial}{\partial u}\left(M_{(i)}\frac{\partial x^i}{\partial v}\right) = 0,$$

hence, again with reference to Figure 1:

$$(6.12) \qquad \left(M_{(i)}\frac{\partial x^i}{\partial v}\right)_{A_0} = \left(M_{(i)}\frac{\partial x^i}{\partial v}\right)_A.$$

By the same argument as in the previous Section:

$$(6.13) \qquad \left(M_{(i)}u^i\right)_{A_0} ds_0 = \left(M_{(i)}u^i\right)_A ds,$$

but, because

$$(6.14) \qquad M_{(i)}u^i = \left[g_{ik} + \left(1 - \frac{1}{\epsilon\mu}\right)u_i u_k\right]M^i u^k = \frac{M_i u^i}{\epsilon\mu},$$

instead of (5.14) one finds

$$(6.15) \qquad \left(\frac{M_i u^i}{\epsilon\mu}\right)_{A_0} ds_0 = \left(\frac{M_i u^i}{\epsilon\mu}\right)_A ds.$$

Since equation (5.16) remains unaltered, instead of (5.17) one eventually obtains

$$(6.16) \qquad \left(\frac{E}{\nu\epsilon\mu}\right)_{A_0} = \left(\frac{E}{\nu\epsilon\mu}\right)_A.$$

## 7. SPACELIKE PHOTONS IN NONDISPERSIVE, ISOTROPIC DIELECTRICS

The photon that we have found through the reduction to the vacuum has the unquestionable virtue of being a timelike entity for which a rest system exists and, as far as a homogeneous medium is considered, it exhibits the proportionality of energy and frequency that a good-natured photon is supposed to possess. But, as soon as a nonhomogeneous medium is considered, the proportionality disappears, and we are left wondering whether we did some wrong, or whether there is some other opportunity that we have missed.

The Hamiltonian point of view tells us that the latter is the case. In fact, with (4.8) there is an ambiguity in the choice of the momentum-energy vector, that is not apparent in the vacuum case. Let us consider the first of Hamilton's equations (2.6). In our case it reads:

$$(7.1) \qquad \frac{\mathrm{d}x^i}{\mathrm{d}u} = \sigma^{ik} f_{,k}.$$

Let

$$(7.2) \qquad p_k = \theta f_{,k}$$

be the covariant version of the momentum-energy vector; it conforms to the usual Hamiltonian prescription for material particles. Then

$$(7.3) \qquad M^i = \sigma^{ik} p_k.$$

In the vacuum case, $M^i$ and $p_k$ are the contravariant and the covariant forms of one and the same entity, whose norm is null. When $\epsilon\mu > 1$, $M^i$ and $p_i$ are expressions of different entities: the first one is a timelike vector, while the second one is spacelike. Let us assume that not $M^i$, but $p_i$ is the momentum-energy of the photon, by definition the one that is exchanged with an atom in the absorption and emission processes. Instead of (5.4) we shall postulate

$$(7.4) \qquad m_0' u_i' = m_0 u_i + p_i.$$

If this equation is rewritten in contravariant form and the two forms are multiplied term by term one obtains

$$(7.5) \qquad -m_0'^2 = -m_0^2 + 2m_0 p_i u^i + p_i p^i.$$

For optical processes one can neglect the last term and define

$$(7.6) \qquad \bar{E} = -p_i u^i$$

as the energy exchanged by the photon. Due to (7.3) one writes

$$(7.7) \qquad M_i u^i = u_k \sigma^{kl} p_l = u_k \left[ g^{kl} - (\epsilon\mu - 1) u^k u^l \right] p_l = \epsilon\mu p_i u^i,$$

hence, instead of the relation (6.16), $\bar{E}$ fulfils

$$(7.8) \qquad \left( \frac{\bar{E}}{\nu} \right)_{A_0} = \left( \frac{\bar{E}}{\nu} \right)_A.$$

## 8. Conclusion

While general relativity in vacuo contains a satisfactory model of the photon as a null particle whose world-line lies in a null wave, and for which the proportionality between energy and frequency holds, the situation is much less encouraging if one attempts defining, by general relativistic methods, a photon in nondispersive, isotropic dielectrics.

The Hamiltonian viewpoint shows that in such dielectrics the photon should be a pointlike entity associated to the ray of geometrical optics, that in its turn is permanently associated to a given phase wave, but it shows

also that one must choose between two evils: either maintaining the proportionality between energy and frequency at the cost of associating a spacelike momentum to an entity that travels along a timelike ray, or insisting on the natural association of a timelike momentum to the timelike ray at the cost of abandoning, when the dielectric is nonhomogeneous, the proportionality of energy and frequency. The old dilemma, born with the electromagnetic energy tensors of Minkowski [7] and Abraham [6] and still awaiting for an experimental resolution [12], thanks to the methods of Gordon [3] and Synge [2] appears here in a novel form.

## References

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