EINSTEIN'S UNIFIED FIELD THEORY PREDICTS THE EQUILIBRIUM POSITIONS OF N WIRES RUN BY STEADY ELECTRIC CURRENTS

SALVATORE ANTOCI

Abstract. A particular exact solution of Einstein's Hermitian theory of relativity is examined, after recalling that there is merit in adding phenomenological sources to the theory, and in choosing the metric like it was done long ago by Kurşunoğlu and Hély. It is shown by intrinsic arguments, relying on the properties of the chosen metric manifold, that the solution describes in Einstein's theory the field of n thin parallel wires at rest, run by steady electric currents, and predicts their equilibrium positions through the injunction that the metric must display cylindrical symmetry in the infinitesimal neighbourhood of each wire. In the weak field limit the equilibrium positions coincide with the ones prescribed by Maxwell's electrodynamics.

1. Introduction

The theory of the nonsymmetric field, after an early attempt by Einstein [1], was separately developed in the same years, but starting from different viewpoints, both by Einstein [2, 3, 4] and by Schrödinger [5, 6, 7, 8]. Both of them thought that the theory had to be some natural generalization of a successful predecessor, the general theory of relativity of 1915. But, while Einstein decided to deal with the nonsymmetric fundamental tensor and with the nonsymmetric affine connection as independent entities, Schrödinger's preference was for the purely affine approach. Remarkably enough, they ended up with what soon appeared, from a pragmatic standpoint, like two versions of one and the same theory, because Schrödinger's "final affine field laws" just look like the laws of Einstein's "generalized theory of gravitation", to which a cosmological term is appended.

It was a conviction both by Einstein and by Schrödinger that, since the theories had to be the completion of the theory of 1915, neither a phenomenological energy tensor nor phenomenological currents had to be added at the right-hand sides of the field equations. However, exact solutions complying with this injunction were never found: exact spherically symmetric solutions displayed singularities [9, 10], i.e. were useless for the envisaged program. Approximate calculations by Callaway [11], although later shown to be incomplete by Narlikar and Rao [12] and by Treder [13], spread the conviction that the theory did not contain the electromagnetic interaction; moreover, they too allowed for singularities. In any case, the formal simplicity of the sets of equations proposed both by Einstein and by Schrödinger did not find a counterpart in an equally simple and satisfactory physical interpretation, and the interest aroused by their endeavour began to fade. Already in the 1954/55 report to the Dublin Institute for Advanced Studies, a disappointed Erwin Schrödinger wrote: "It is a disconcerting situation that ten years endeavour of competent theorists has not yielded even a plausible glimpse of Coulomb's law."[14].

An alternative viewpoint on those equations was however possible, in some way bolstered by the ubiquitous presence of singularities in the solutions found, by the very form of the contracted Bianchi identities, and by the strict similarity of the new equations to the field equations of 1915. It was expressed [15, 16] in 1954 by Hély, who could avail in his attempt of previous findings by Kurşunoğlu [17, 18] and by Lichnerowicz [19] on the choice of the metric. According to Hély, in the new theory of Einstein phenomenological sources, in the form of a symmetric energy tensor and of a conserved four-current, had to be appended respectively at the right-hand sides of the field equations (A.4) and (A.5), given in Appendix (A). By pursuing further Hély's proposal, and by relying on a precious finding by Borchsenius [20], the way for appending sources to all the field equations while keeping the choice of the metric done by Hély was later investigated [21], and is the subject of the next Section.

2. Appending sources to Einstein's unified field theory

On a four-dimensional manifold, let $\mathbf{g}^{ik}$ be a contravariant tensor density with an even part $\mathbf{g}^{(ik)}$ and an alternating one $\mathbf{g}^{[ik]}$:

$$\mathbf{g}^{ik} = \mathbf{g}^{(ik)} + \mathbf{g}^{[ik]}, \tag{2.1}$$

and W^i_{kl} be a general affine connection

$$W^i_{kl} = W^i_{(kl)} + W^i_{[kl]}. \tag{2.2}$$

The Riemann curvature tensor built from W^i_{kl}:

$$R^i{}_{klm}(W) = W^i_{kl,m} - W^i_{km,l} - W^i_{al}W^a_{km} + W^i_{am}W^a_{kl}, \tag{2.3}$$

has two distinct contractions, $R_{ik}(W) = R^p{}_{ikp}(W)$ and $A_{ik}(W) = R^p{}_{pik}(W)$ [22]. But the transposed affine connection $\tilde{W}^i_{kl} = W^i_{lk}$ must be considered too: from it, the Riemann curvature tensor $R^i{}_{klm}(\tilde{W})$ and its two contractions $R_{ik}(\tilde{W})$ and $A_{ik}(\tilde{W})$ can be formed as well. We aim at following the pattern of general relativity, which is built from the Lagrangian density $\mathbf{g}^{ik}R_{ik}$, but now any linear combination $\bar{R}_{ik}$ of the four above-mentioned contractions is possible. A good choice [20], for physical reasons that will become apparent later, is

$$\bar{R}_{ik}(W) = R_{ik}(W) + \frac{1}{2}A_{ik}(\tilde{W}). \tag{2.4}$$

Let us provisionally endow the theory with sources in the form of a non-symmetric tensor P_{ik} and of a current density $\mathbf{j}^i$, coupled to $\mathbf{g}^{ik}$ and to the vector $W_i = W^l_{[il]}$ respectively. The Lagrangian density

$$\mathbf{L} = \mathbf{g}^{ik}\bar{R}_{ik}(W) - 8\pi\mathbf{g}^{ik}P_{ik} + \frac{8\pi}{3}W_i\mathbf{j}^i \tag{2.5}$$

is thus arrived at. By performing independent variations of the action $\int \mathbf{L}d\Omega$ with respect to W^p_{qr} and to $\mathbf{g}^{ik}$ with suitable boundary conditions we obtain the field equations

$$\begin{aligned} &-\mathbf{g}^{qr}_{,p} + \delta^r_p\mathbf{g}^{(sq)}_{,s} - \mathbf{g}^{sr}W^q_{sp} - \mathbf{g}^{qs}W^r_{ps} \\ &+\delta^r_p\mathbf{g}^{st}W^q_{st} + \mathbf{g}^{qr}W^t_{pt} = \frac{4\pi}{3}(\mathbf{j}^r\delta^q_p - \mathbf{j}^q\delta^r_p) \end{aligned} \tag{2.6}$$

and

$$\bar{R}_{ik}(W) = 8\pi P_{ik}. \tag{2.7}$$

By contracting eq. (2.6) with respect to q and p we get

$$\mathbf{g}^{[is]}_{,s} = 4\pi\mathbf{j}^i. \tag{2.8}$$

The very finding of this physically welcome equation entails however that we cannot determine the affine connection W^i_{kl} uniquely in terms of $\mathbf{g}^{ik}$: (2.6) is invariant under the projective transformation ${W'}^i_{kl} = W^i_{kl} + \delta^i_k\lambda_l$, where λ_l is an arbitrary vector field. Moreover eq. (2.7) is invariant under the transformation

$${W'}^i_{kl} = W^i_{kl} + \delta^i_k\mu_{,l} \tag{2.9}$$

where μ is an arbitrary scalar. By following Schrödinger [7, 22], we write

$$W^i_{kl} = \Gamma^i_{kl} - \frac{2}{3}\delta^i_k W_l, \tag{2.10}$$

where Γ^i_{kl} is another affine connection, by definition constrained to yield $\Gamma^l_{[il]=0}$. Then eq. (2.6) becomes

$$\mathbf{g}^{qr}_{,p} + \mathbf{g}^{sr}\Gamma^q_{sp} + \mathbf{g}^{qs}\Gamma^r_{ps} - \mathbf{g}^{qr}\Gamma^t_{(pt)} = \frac{4\pi}{3}(\mathbf{j}^q\delta^r_p - \mathbf{j}^r\delta^q_p) \tag{2.11}$$

that allows one to determine Γ^i_{kl} uniquely, under very general conditions [23], in terms of $\mathbf{g}^{ik}$. When eq. (2.10) is substituted in eq. (2.7), the latter comes to read

$$\bar{R}_{(ik)}(\Gamma) = 8\pi P_{(ik)} \tag{2.12}$$

$$\bar{R}_{[ik]}(\Gamma) = 8\pi P_{[ik]} - \frac{1}{3}(W_{i,k} - W_{k,i}) \tag{2.13}$$

after splitting the even and the alternating parts. Wherever the source term is nonvanishing, a field equation loses its rôle, and becomes a definition of some property of matter in terms of geometrical entities; it is quite obvious that such a definition must be unique. This occurs with eqs. (2.8), (2.11) and (2.12), but it does not happen for eq. (2.13). This equation only prescribes that $\bar{R}_{[ik]}(\Gamma)-8\pi P_{[ik]}$ is the curl of the arbitrary vector $W_i/3$; it is therefore equivalent to the four equations

$$\bar{R}_{[ik],l}(\Gamma)+\bar{R}_{[kl],i}(\Gamma)+\bar{R}_{[li],k}(\Gamma)=8\pi\{P_{[ik],l}+P_{[kl],i}+P_{[li],k}\}, \tag{2.14}$$

that cannot specify $P_{[ik]}$ uniquely. We therefore scrap the redundant tensor $P_{[ik]}$, like we scrapped the redundant affine connection W^i_{kl} of eq. (2.6), and assume that matter is described by the symmetric tensor $P_{(ik)}$, by the conserved current density $\mathbf{j}^i$ and by the conserved current

$$K_{ikl}=\frac{1}{8\pi}\{\bar{R}_{[ik],l}+\bar{R}_{[kl],i}+\bar{R}_{[li],k}\}. \tag{2.15}$$

The general relativity of 1915, to which the present theory reduces when $\mathbf{g}^{[ik]}=0$, suggests rewriting eq. (2.12) as

$$\bar{R}_{(ik)}(\Gamma)=8\pi(T_{ik}-\frac{1}{2}s_{ik}s^{pq}T_{pq}) \tag{2.16}$$

where $s_{ik}=s_{ki}$ is the still unchosen metric tensor of the theory, $s^{il}s_{kl}=\delta^i_k$, and the symmetric tensor T_{ik} will act as energy tensor.

When sources are vanishing, equations (2.11), (2.16), (2.8) and (2.15) reduce to the original equations of Einstein's unified field theory, reported in Appendix (A), because then $\bar{R}_{ik}(\Gamma)$=$R_{ik}(\Gamma)$; moreover they enjoy the property of transposition invariance also when the sources are nonvanishing. If $\mathbf{g}^{ik}$, Γ^i_{kl}, $\bar{R}_{ik}(\Gamma)$ represent a solution with the sources T_{ik}, $\mathbf{j}^i$ and K_{ikl}, the transposed quantities $\tilde{\mathbf{g}}^{ik}=\mathbf{g}^{ki}$, $\tilde{\Gamma}^i_{kl}=\Gamma^i_{lk}$ and $\bar{R}_{ik}(\tilde{\Gamma})$= $\bar{R}_{ki}(\Gamma)$ represent another solution, endowed with the sources $\tilde{T}_{ik}=T_{ik}$, $\tilde{\mathbf{j}}^i=-\mathbf{j}^i$ and $\tilde{K}_{ikl}=-K_{ikl}$. Such a physically desirable outcome is a consequence of the choice made [20] for $\bar{R}_{ik}$. These equations intimate that Einstein's unified field theory with sources should be interpreted like a gravoelectrodynamics in a polarizable continuum, allowing for both electric and magnetic currents. The study of the conservation identities confirms the idea [21] and provides at the same time the identification of the metric tensor s_{ik}. Let us consider the invariant integral

$$I=\int\left[\mathbf{g}^{ik}\bar{R}_{ik}(W)+\frac{8\pi}{3}W_i\mathbf{j}^i\right]d\Omega. \tag{2.17}$$

From it, when eq. (2.6) is assumed to hold, by means of an infinitesimal coordinate transformation we get the four identities

$$\begin{aligned}&-(\mathbf{g}^{is}\bar{R}_{ik}(W)+\mathbf{g}^{si}\bar{R}_{ki}(W))_{,s}+\mathbf{g}^{pq}\bar{R}_{pq,k}(W)\\&\qquad+\frac{8\pi}{3}\mathbf{j}^i(W_{i,k}-W_{k,i})=0.\end{aligned} \tag{2.18}$$

This equation can be rewritten as

$$\begin{aligned}&-2(\mathbf{g}^{(is)}\bar{R}_{(ik)}(\Gamma))_{,s}+\mathbf{g}^{(pq)}\bar{R}_{(pq),k}(\Gamma)\\&=2\mathbf{g}^{[is]}_{,s}\bar{R}_{[ik]}(\Gamma)+\mathbf{g}^{[is]}\left\{\bar{R}_{[ik],s}(\Gamma)+\bar{R}_{[ks],i}(\Gamma)+\bar{R}_{[si],k}(\Gamma)\right\}\end{aligned} \tag{2.19}$$

where the redundant variable W^i_{kl} no longer appears. We remind of eq. (2.16) and, by following Kurşunoğlu [17, 18] and Hély [15, 16], we assume that the metric tensor is defined by the equation

$$\sqrt{-s}s^{ik} = \mathbf{g}^{(ik)}, \tag{2.20}$$

where $s = \det(s_{ik})$; we shall use henceforth s^{ik} and s_{ik} to raise and lower indices, $\sqrt{-s}$ to produce tensor densities out of tensors. We define then

$$\mathbf{T}^{ik} = \sqrt{-s}s^{ip}s^{kq}T_{pq} \tag{2.21}$$

and the weak identities (2.19), when all the field equations hold, will take the form

$$\mathbf{T}^{ls}_{;s} = \frac{1}{2}s^{lk}(\mathbf{j}^i\bar{R}_{[ki]}(\Gamma) + K_{iks}\mathbf{g}^{[si]}), \tag{2.22}$$

where the semicolon means covariant derivative with respect to the Christoffel affine connection

$$\left\{ {i \atop k\ l} \right\} = \frac{1}{2}s^{im}(s_{mk,l} + s_{ml,k} - s_{kl,m}) \tag{2.23}$$

built with s_{ik}. The previous impression is strengthened by eq. (2.22): the theory, built in terms of a non-Riemannian geometry, appears to entail a gravoelectrodynamics in a dynamically polarized Riemannian spacetime, for which s_{ik} is the metric, where the two conserved currents $\mathbf{j}^i$ and K_{iks} are coupled à la Lorentz to $\bar{R}_{[ki]}$ and to $\mathbf{g}^{[si]}$ respectively. Two versions of this gravoelectrodynamics are possible, according to whether $\mathbf{g}^{ik}$ is chosen to be either a real nonsymmetric or a complex Hermitian tensor density. The constitutive relation between electromagnetic inductions and fields is governed by the field equations in a quite novel and subtle way: the link between $\mathbf{g}^{[ik]}$ and $\bar{R}_{[ik]}$ is not the simple algebraic one usually attributed to the vacuum, with some metric that raises or lowers indices, and builds densities from tensors. It is a differential one, and a glance to the field equations suffices to become convinced that understanding its properties is impossible without first finding and perusing the exact solutions of the theory.

This may seem a hopeless endeavour. However, a class of exact solutions intrinsically depending on three coordinates has been found; the method for obtaining them from vacuum solutions of the general relativity of 1915 is described in Appendix (B). Some of these solutions happen to assume physical meaning when source terms are appended to the field equations in the way described in this Section. In particular, two static solutions built in this way has been interpreted. One of them happens to describe the general electrostatic field of n localised charges ([24]) built by the four-current density $\mathbf{j}^i$ defined by equation (2.8). As expected [25], the nonlinearity of the theory rules the singular behaviour of the metric field s_{ik} in the proximity of each charge. It rules it in such a way that, with all the approximation needed to comply with the experimental facts, the charges happen to be pointlike in the metric sense and endowed with spherically symmetric neighbourhoods only when they occupy mutual positions that correspond to the ones dictated by Coulomb's law.

Another static, axially symmetric solution [26, 27] displays instead n aligned pole sources built with the four-current K_{ikl} defined by equation (2.15). The differential constitutive relation between $\mathbf{g}^{[ik]}$ and $\bar{R}_{[ik]}$, however, avoids the unphysical result that these charges behave like magnetic monopoles would do, if they were allowed for in the so-called Einstein-Maxwell theory. In fact, the study of a particular solution endowed with three such aligned charges shows that these pointlike charges interact with forces not depending on their mutual distance. In the Hermitian version of the theory, charges with opposite signs happen to mutually attract, hence they are permanently confined entities, like it was already shown by Treder [13] with approximate calculations based both on the E.I.H. [28, 25] and on the Papapetrou [29] method.

In the present paper the behaviour of a solution displaying n steady currents built with $\mathbf{j}^i$ and running on parallel wires is instead investigated, by using s_{ik} as metric tensor.

3. The equilibrium conditions of steady electric currents running on n parallel wires

This solution belongs to the class described in Appendix (B); in it, only one of equations (2.8) is not trivially satisfied, and reads

$$\mathbf{g}^{[3s]}_{\ ,s} = i\left(\sqrt{-h}h^{\varrho\sigma}\xi_{,\sigma}\right)_{,\varrho} = 4\pi\mathbf{j}^3(x^\lambda). \tag{3.1}$$

Like the electrostatic solution, this one too is obtained by assuming that h_{ik} has the Minkowski form

$$h_{ik} = \mathrm{diag}(-1,-1,-1,1), \tag{3.2}$$

with respect to the coordinates $x^1 = x$, $x^2 = y$, $x^3 = z$, $x^4 = t$. In these coordinates its fundamental form g_{ik}, defined by (B.2), reads:

$$g_{ik} = \begin{pmatrix} -1 & 0 & e & 0 \\ 0 & -1 & f & 0 \\ -e & -f & v & c \\ 0 & 0 & -c & 1 \end{pmatrix}, \tag{3.3}$$

with

$$v = -1 - c^2 + e^2 + f^2 \tag{3.4}$$

and

$$e = i\xi_{,x},\ f = i\xi_{,y},\ c = -i\xi_{,t},\quad i = \sqrt{-1},\quad \xi_{,xx} + \xi_{,yy} - \xi_{,tt} = 0. \tag{3.5}$$

Let us consider the particular, static solution for which

$$\xi = \sum_{k=1}^{n} l_k \ln q_k \tag{3.6}$$

where

$$q_k = \left[(x-x_k)^2 + (y-y_k)^2\right]^{1/2}, \tag{3.7}$$

and l_k, x_k, y_k are arbitrary real constants. Then one finds

$$e = i\sum_{k=1}^{n} l_k \frac{x-x_k}{q_k^2},\ f = i\sum_{k=1}^{n} l_k \frac{y-y_k}{q_k^2},\ c = 0, \tag{3.8}$$

$$v = -1 - \sum_{k=1}^{n} \frac{l_k^2}{q_k^2} - \left[\sum_{k,k'=1}^{n} l_k l_{k'} \frac{(x-x_k)(x-x_{k'}) + (y-y_k)(y-y_{k'})}{q_k^2 q_{k'}^2}\right]_{k\neq k'}. \tag{3.9}$$

We note in passing that, despite its rôle as component of the fundamental tensor, $v = -1 + \frac{1}{2} g_{[ik]} g^{[ik]}$ is an invariant quantity.

In this solution the vacuum field equation $\mathbf{g}^{[is]}_{\ ,s} = 0$ is satisfied everywhere, with the exception of the positions $x = x_k$, $y = y_k$, $k = 1, .., n$, of the wires in the representative space, while, due to the additional conditions (B.3), the additional invariant equation

$$g_{[ik],l} + g_{[kl],i} + g_{[li],k} = 0, \tag{3.10}$$

is fulfilled everywhere. As far as the skew fields are concerned, we are therefore inclined to interpret physically this solution as the field produced by n steady electric currents running along thin wires drawn parallel to the z coordinate axis. But in order to do so, we need the proof, coming from the symmetric field s_{ik}, that the wires are indeed thin in the metric sense, and that the metric is endowed with cylindrical symmetry in the infinitesimal neighbourhood of each wire. The latter property is required for the wires to be in static equilibrium, in keeping with a deep intuition present in [25]. To provide this proof, we shall examine the square $\mathrm{d}s^2$ of the interval, defined by (B.13), that in the present case happens to read

$$\mathrm{d}s^2 = \sqrt{-v}(\mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2) + \frac{(\mathrm{d}\xi)^2}{\sqrt{-v}}. \tag{3.11}$$

The first term of the interval is conformally flat. Due to the existence of two Killing vectors, respectively along t and along z, and both orthogonal to each x, y two-surface, it suffices that we examine the problem on a given x, y two-surface of the manifold. We shall prove that the sections of the wires by the given two-surface are pointlike in the metrical sense, and we shall require that s_{ik}, in the infinitesimal neighbourhood of each point $x = x_k$, $y = y_k$ of that two-surface, is endowed with invariance under rotation around these points.

The first question is soon answered. In fact, from the behaviour of v, defined by (3.9), in an infinitesimal neighbourhood (in the "Bildraum" sense) of $x = x_k$, $y = y_k$, one gathers that the first term

$$\mathrm{d}s_1^2 = \sqrt{-v}(\mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2) \tag{3.12}$$

will vanish like q_k when $q_k \to 0$. The second term

$$\mathrm{d}s_2^2 = \frac{(\mathrm{d}\xi)^2}{\sqrt{-v}} \tag{3.13}$$

is now examined in the chosen neighbourhood. Due to the definition (3.6) of ξ, the numerator $(\mathrm{d}\xi)^2$ will keep there a finite value, while the denominator $\sqrt{-v}$ diverges like q_k^{-1} when $q_k \to 0$. Therefore the term (3.13) too shall vanish like q_k in the considered, infinitesimal neighbourhood. One concludes that, when s_{ik} is the metric tensor, the n parallel wires of this solution will be infinitely thin in the metric sense for any physically reasonable choice of their mutual positions.

We examine now the second question, whether and under what conditions the metric field s_{ik} will exhibit rotational symmetry in the infinitesimal neighbourhood of each point $x = x_k$, $y = y_k$ of the considered two-surface. Let us imagine approaching the k-th wire along the line defined by

$$x - x_k = n_x q_k, \quad y - y_k = n_y q_k, \tag{3.14}$$

where n_x and n_y are constants for which $n_x^2 + n_y^2 = 1$, but otherwise arbitrary. To study $\mathrm{d}s_1^2$ we need evaluating $\sqrt{-v}$ in the infinitesimal neighbourhood of $x = x_k$, $y = y_k$. Let us call this quantity $\{\sqrt{-v}\}_{(k)}$. One first extracts from the root the common factor l_k/q_k, and then subjects the other factor to a Taylor's expansion truncated to terms that vanish like q_k when $q_k \to 0$. Higher order terms will not influence the final result. One then writes:

$$\{\sqrt{-v}\}_{(k)} \simeq \frac{l_k}{q_k}\left[1 + \frac{q_k}{2l_k}\sum_{k' \neq k}^{n} l_{k'} \frac{n_x(x_k - x_{k'}) + n_y(y_k - y_{k'})}{d_{kk'}^2}\right], \tag{3.15}$$

where

$$d_{kk'}^2 = (x_k - x_{k'})^2 + (y_k - y_{k'})^2. \tag{3.16}$$

If the term with the summation symbol were lacking, $\{\sqrt{-v}\}_{(k)}$ would display rotational symmetry, in the "Bildraum" sense, in the infinitesimal neighbourhood of $x = x_k$, $y = y_k$. Since n_x, n_y fulfill $n_x^2 + n_y^2 = 1$, but are otherwise arbitrary, the mentioned symmetry only occurs when the conditions

$$\sum_{k' \neq k}^{n} l_{k'} \frac{x_k - x_{k'}}{d_{kk'}^2} = 0, \quad \sum_{k' \neq k}^{n} l_{k'} \frac{y_k - y_{k'}}{d_{kk'}^2} = 0, \tag{3.17}$$

are severally satisfied. When this occurs the interval $\mathrm{d}s_1^2$, given by (3.12), will be endowed with rotational symmetry in an infinitesimal neighbourhood surrounding $x = x_k$, $y = y_k$ in an intrinsic, geometric sense.

We examine now $\mathrm{d}s_2^2$, defined by (3.13), in the infinitesimal neighbourhood of $x = x_k$, $y = y_k$ of the considered x, y two-surface. Since ξ is defined by (3.6), one can write

$$\{\mathrm{d}\xi\}_{(k)} = \{\xi_{,x}\mathrm{d}x + \xi_{,y}\mathrm{d}y\}_{(k)} \tag{3.18}$$

$$\simeq \left[\frac{l_k n_x}{q_k} + \sum_{k'\neq k}^{n} l_{k'} \frac{x_k - x_{k'}}{d_{kk'}^2}\right]\mathrm{d}x + \left[\frac{l_k n_y}{q_k} + \sum_{k'\neq k}^{n} l_{k'} \frac{y_k - y_{k'}}{d_{kk'}^2}\right]\mathrm{d}y,$$

by neglecting all the terms that vanish when $q_k \to 0$, because they will not influence the final result. To calculate $\left\{(\sqrt{-v})^{-1}\right\}_{(k)}$, let us extract from $(\sqrt{-v})^{-1}$ the factor q_k/l_k, and then expand the other factor in Taylor's series around $x = x_k$, $y = y_k$. We truncate the expansion at the term linear in q_k, because higher order terms do not influence the final outcome. Therefore we write

$$\left\{(\sqrt{-v})^{-1}\right\}_{(k)} \simeq \frac{q_k}{l_k}\left[1 - \frac{q_k}{2l_k}\sum_{k'\neq k}^{n} l_{k'} \frac{n_x(x_k - x_{k'}) + n_y(y_k - y_{k'})}{d_{kk'}^2}\right], \tag{3.19}$$

and calculate $\left\{(\mathrm{d}\xi)^2/\sqrt{-v}\right\}_{(k)}$ from (3.18) and (3.19). One finds that $\mathrm{d}s_2^2$ in general does not exhibit rotational symmetry in the infinitesimal neighbourhood of $x = x_k$, $y = y_k$. Only if the conditions (3.17) are imposed, $\left\{(\mathrm{d}\xi)^2/\sqrt{-v}\right\}_{(k)}$ comes to read

$$\left\{(\mathrm{d}\xi)^2/\sqrt{-v}\right\}_{(k)} \simeq \frac{l_k}{q_k}\left(n_x^2\mathrm{d}x^2 + n_y^2\mathrm{d}y^2 + 2n_x n_y \mathrm{d}x\mathrm{d}y\right), \tag{3.20}$$

i.e. it defines a two-dimensional interval endowed with rotational symmetry in the intrinsic, geometric sense. Therefore, only if the conditions (3.17) are imposed, both $\mathrm{d}s_1^2$ and $\mathrm{d}s_2^2$ become endowed with rotational symmetry, in an intrinsic, geometric sense, in the infinitesimal neighbourhoods around each one of the points $x = x_k$, $y = y_k$, and the same property will be exhibited by the interval $\mathrm{d}s^2$, defined by (3.11).

4. Conclusion

The equations (3.1) together with (3.8), and (3.10), as well as the general formulation of Section (2), already led to think that the present solution physically describes, in Einstein's unified field theory, the field originated by steady electric currents running on n parallel wires. The scrutiny of the interval (3.11) confirms this interpretation. In fact, when the metric is s_{ik}, the parallel wires of the "Bildraum" turn out to be infinitely thin parallel wires in the metric sense. Moreover, the infinitesimal neighbourhood of each of these wires happens to be endowed with cylindrical symmetry only when the conditions (3.17), with their distinct flavour of *dèjá vu*, are satisfied. The equilibrium conditions for n parallel wires run by steady currents in Maxwell's electrodynamics are just written in that way. It is true that we shall not be deceived by retrieving their exact replica in Einstein's unified field theory with sources, because this is just an accidental occurrence due to the particular coordinates adopted when solving the field equations. When measured along geodesics built with the metric s_{ik}, distances and angles are different from the ones that would prevail if the conditions (3.17) were read as if they would hold in a Minkowski metric. But there is no doubt that in the weak field limit the particular exact solutions of both theories describe one and the same physical reality.

As stressed long ago by Einstein in the Introduction of both ([28]) and ([25]), there is one distinct advantage in working with such nonlinear theories as the general relativity of 1915, or its nonsymmetric generalization. While in the linear physics of, say, Maxwell's electrodynamics, the field equations and the equations of motion need to be separately postulated, this is no longer the case in, e.g., the Hermitian theory with sources. In the latter theory, all what is needed is solving the field equations. From the very solution one learns the equations of motion, by just imposing symmetry conditions on the metric around the singularities that are used to represent the physical objects. In the case of a static manifold, one learns the equilibrium conditions of such objects, like it has been shown, once more, through the exact solution of the previous Section.

Appendix A. Hermitian field equations without sources

We consider here Einstein's unified field theory in the Hermitian version ([3]). A given geometric quantity [30] will be called hereafter Hermitian with respect to the indices i and k, both either covariant or contravariant, if the part of the quantity that is symmetric with respect to i and k is real, while the part that is antisymmetric is purely imaginary. Let us consider the Hermitian fundamental form $g_{ik} = g_{(ik)} + g_{[ik]}$ and the affine connection $\Gamma^i_{kl} = \Gamma^i_{(kl)} + \Gamma^i_{[kl]}$, Hermitian with respect to the lower indices; both entities depend on the real coordinates x^i, with i running from 1 to 4. We define also the Hermitian contravariant tensor g^{ik} by the relation

$$g^{il} g_{kl} = \delta^i_k, \tag{A.1}$$

and the contravariant tensor density $\mathbf{g}^{ik} = (-g)^{1/2} g^{ik}$, where $g \equiv \det(g_{ik})$ is a real quantity. Then the field equations of Einstein's unified field theory in the complex Hermitian form [3] read

$$g_{ik,l} - g_{nk}\Gamma^n_{il} - g_{in}\Gamma^n_{lk} = 0, \tag{A.2}$$

$$\mathbf{g}^{[is]}_{\ ,s} = 0, \tag{A.3}$$

$$R_{(ik)}(\Gamma) = 0, \tag{A.4}$$

$$R_{[ik],l}(\Gamma) + R_{[kl],i}(\Gamma) + R_{[li],k}(\Gamma) = 0; \tag{A.5}$$

$R_{ik}(\Gamma)$ is the Hermitian Ricci tensor

$$R_{ik}(\Gamma) = \Gamma^a_{ik,a} - \Gamma^a_{ia,k} - \Gamma^a_{ib}\Gamma^b_{ak} + \Gamma^a_{ik}\Gamma^b_{ab}. \tag{A.6}$$

Appendix B. Solutions depending on three coordinates [31]

We assume that Greek indices take the values 1,2 and 4, while Latin indices run from 1 to 4. Let the real symmetric tensor h_{ik} be the metric for a vacuum solution to the field equations of the general relativity of 1915, which depends on the three co-ordinates x^λ, not necessarily all spatial in character, and for which $h_{\lambda 3} = 0$. We consider also an antisymmetric purely imaginary tensor a_{ik}, which depends too only on the co-ordinates x^λ, and we assume that its only nonvanishing components are $a_{\mu 3} = -a_{3\mu}$. Then we form the mixed tensor

$$\alpha_i{}^k = a_{il}h^{kl} = -\alpha^k{}_i, \tag{B.1}$$

where h^{ik} is the inverse of h_{ik}, and we define the Hermitian fundamental form g_{ik} as follows:

$$\begin{aligned} g_{\lambda\mu} &= h_{\lambda\mu}, \\ g_{3\mu} &= \alpha_3{}^\nu h_{\mu\nu}, \\ g_{33} &= h_{33} - \alpha_3{}^\mu \alpha_3{}^\nu h_{\mu\nu}. \end{aligned} \tag{B.2}$$

When the three additional conditions

$$\alpha^3{}_{\mu,\lambda} - \alpha^3{}_{\lambda,\mu} = 0 \tag{B.3}$$

are fulfilled, the affine connection Γ^i_{kl} which solves eqs. (A.2) has the non-vanishing components

$$\begin{aligned} \Gamma^\lambda_{(\mu\nu)} &= \left\{ {\lambda \atop \mu\ \nu} \right\}_{(h)}, \\ \Gamma^\lambda_{[3\nu]} &= \alpha_3{}^\lambda{}_{,\nu} - \left\{ {3 \atop 3\ \nu} \right\}_{(h)} \alpha_3{}^\lambda + \left\{ {\lambda \atop \rho\ \nu} \right\}_{(h)} \alpha_3{}^\rho, \\ \Gamma^3_{(3\nu)} &= \left\{ {3 \atop 3\ \nu} \right\}_{(h)}, \\ \Gamma^\lambda_{33} &= \left\{ {\lambda \atop 3\ 3} \right\}_{(h)} - \alpha_3{}^\nu \left(\Gamma^\lambda_{[3\nu]} - \alpha_3{}^\lambda \Gamma^3_{(3\nu)} \right); \end{aligned} \tag{B.4}$$

we indicate with $\left\{ {i \atop k\ l} \right\}_{(h)}$ the Christoffel connection built with h_{ik}. We form now the Ricci tensor (A.6). When eqs. (A.3), i.e., in our case, the single equation

$$(\sqrt{-h}\ \alpha_3{}^\lambda h^{33})_{,\lambda} = 0, \tag{B.5}$$

and the additional conditions, expressed by eqs. (B.3), are satisfied, the components of $R_{ik}(\Gamma)$ can be written as

$$\begin{aligned} R_{\lambda\mu} &= H_{\lambda\mu}, \\ R_{3\mu} &= \alpha_3{}^\nu H_{\mu\nu} + \left(\alpha_3{}^\nu \left\{ {3 \atop 3\ \nu} \right\}_{(h)} \right)_{,\mu}, \\ R_{33} &= H_{33} - \alpha_3{}^\mu \alpha_3{}^\nu H_{\mu\nu}, \end{aligned} \tag{B.6}$$

where H_{ik} is the Ricci tensor built with $\left\{ {i \atop k\ l} \right\}_{(h)}$. H_{ik} is zero when h_{ik} is a vacuum solution of the field equations of general relativity, as supposed;

therefore, when eqs. (B.3) and (B.5) hold, the Ricci tensor, defined by eqs. (B.6), satisfies eqs. (A.4) and (A.5) of the Hermitian theory of relativity.

The task of solving equations (A.2)-(A.5) reduces, under the circumstances considered here, to the simpler task of solving eqs. (B.3) and (B.5) for a given h_{ik}.[1]

Let us assume, like in Section (2), that the metric tensor is defined by the equation

$$\sqrt{-s}s^{ik} = \mathbf{g}^{(ik)}, \tag{B.7}$$

where $s^{il}s_{kl} = \delta^i_k$ and $s = \det(s_{ik})$. When the fundamental tensor g_{ik} has the form (B.2) it is

$$\sqrt{-g} = \sqrt{-h}, \tag{B.8}$$

where $h \equiv \det(h_{ik})$, and

$$\det\left(g^{(ik)}\right) = \frac{1 - g^{3\tau}g_{3\tau}}{h}. \tag{B.9}$$

Therefore

$$\sqrt{-s} = \sqrt{-h}\left(1 - g^{3\tau}g_{3\tau}\right)^{1/2}, \tag{B.10}$$

hence

$$s^{ik} = g^{(ik)}\left(1 - g^{3\tau}g_{3\tau}\right)^{-1/2}. \tag{B.11}$$

The nonvanishing components of s_{ik} then read

$$\begin{aligned} s_{\lambda\mu} &= \left(1 - g^{3\tau}g_{3\tau}\right)^{1/2} h_{\lambda\mu} + \left(1 - g^{3\tau}g_{3\tau}\right)^{-1/2} h_{33}\alpha^3{}_\lambda\alpha^3{}_\mu, \\ s_{33} &= \left(1 - g^{3\tau}g_{3\tau}\right)^{1/2} h_{33}, \end{aligned} \tag{B.12}$$

and the square of the interval $\mathrm{d}s^2 = s_{ik}\mathrm{d}x^i\mathrm{d}x^k$ eventually comes to read

$$\mathrm{d}s^2 = \left(1 - g^{3\tau}g_{3\tau}\right)^{1/2} h_{ik}\mathrm{d}x^i\mathrm{d}x^k - \left(1 - g^{3\tau}g_{3\tau}\right)^{-1/2} h_{33}\left(\mathrm{d}\xi\right)^2. \tag{B.13}$$

In keeping with (B.3), we have defined $\alpha^3{}_\mu$ as

$$\alpha^3{}_\mu = i\xi_{,\mu}, \tag{B.14}$$

in terms of the real function $\xi(x^\lambda)$.

[1]This method of solution obviously applies to Schrödinger's purely affine theory [8] too.

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Dipartimento di Fisica "A. Volta" and IPCF of CNR, Pavia, Italy
E-mail address: `Antoci@fisicavolta.unipv.it`