THE PROPAGATION OF WAVES IN EINSTEIN'S UNIFIED FIELD THEORY AS SHOWN BY TWO EXACT SOLUTIONS

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Abstract. The propagation of waves in two space dimensions exhibited by two exact solutions to the field equations of Einstein's unified field theory is investigated under the assumption that the metric s_{ik} is the one already chosen by Kurşunoğlu and by Hély in the years 1952-1954. It is shown that, for both exact solutions, with this choice of the metric the propagation of the waves occurs in the wave zone with the fundamental velocity ($\mathrm{d}s^2 = 0$).

1. Introduction

As soon as the independent, but concurrent efforts by Einstein and by Schrödinger eventually led to the final mathematical formulation of what may be respectively called the metric-affine [1, 2, 3, 4] and the purely affine [5, 6, 7, 8] versions for the nonsymmetric generalization of Einstein's theory of 1915, skilled theoreticians and geometers [1] undertook the difficult task of understanding the physical meaning of the theory through the investigation of its mathematical structure and the search for the solutions, both exact and approximate, to its field equations.

However, progress towards the accomplishment of this task was very slow, if in 1954 Schrödinger still wrote [21] about the very identification of the metric tensor of the theory as an open question [2]. A quite relevant contribution to the identification of the metric tensor in Einstein-Schrödinger unified field theory came from the study of the Cauchy problem done by Lichnerowicz [13, 16]. He succeeded in thorougly analysing the Cauchy problem without solving the unwieldy equation (A.2) explicitly, and proved that the answer to the Cauchy problem is in general unique, unless the surface S, on which the Cauchy data are given, is a characteristic surface, i.e. unless locally

[1]Without pretense of completeness, let us here recall the remarkable achievements by Papapetrou [9], Wyman [10], Kurşunoğlu [11, 12], Lichnerowicz [13, 16], Hély [14, 15], Tonnelat [17], V.V. Narlikar and B.R. Rao [18], Treder [19], Hlavatý [20].

[2]In the last pages of the cited book, he wrote in fact: "We cannot even feel sure whether in the nonsymmetric case the $g_{(ik)}$ or the $\mathbf{g}^{(ik)}$ (or, less likely, the $\mathbf{g}_{(ik)}$ or the $g^{(ik)}$) play the part of the corresponding tensorial entities describing the gravitational field in Einstein's theory". For the definitions of the quoted quantities, see appendix (A) of the present paper.

$S = f(x^i)$, where f fullfils the so-called eikonal equation

$$g^{(ik)} f_{,i} f_{,k} = 0. \tag{1.1}$$

Although, from a purely mathematical perspective, the function f satisfying (1.1) only defines a surface that is unsuitable as a startpoint for the solution of the Cauchy problem, the very fact that (1.1) has just the form of the eikonal equation, i.e. of the equation that stems from d'Alembert equation in the high frequency limit, naturally led to read in it a law of wave propagation, sometimes of a shock wave propagation, ruled by a metric, in the present case by $g^{(ik)}$, or, more precisely, by any tensor conformally related to it. Indeed $\mathbf{g}^{(ik)}$, one of the four candidates considered by Schrödinger [21] for producing a metric, through the stipulation

$$\sqrt{-s} s^{ik} = \mathbf{g}^{(ik)}, \tag{1.2}$$

where $s = \det(s_{ik})$, allows defining a metric tensor s^{ik} that is conformally related to $g^{(ik)}$. Why, among all the tensors that are conformally related to $g^{(ik)}$, just s^{ik} should be chosen as metric, turns out from the quoted results found by Kurşunoğlu [11, 12] and by Hély [14]: with that choice the four identities, that render the field equations (A.2) - (A.5) compatible, assume a very simple and allusive writing. So allusive that, on this basis, Hély [15] decided to disobey the injunction both by Einstein and by Schrödinger, according to which no phenomenological source terms should be appended at the right-hand sides of the field equations (A.2) - (A.5). In that way the conservation identities, that are otherwise empty, appear to assume physical meaning. More recently, while retaining Hély's choice of the metric tensor s_{ik}, and by availing of a crucial finding by Borchsenius [22], Hély's approach was extended [23], by adding phenomenological sources at the right-hand sides of all the field equations, in the form of a symmetric energy-momentum tensor, and of two conserved four-current densities. The way for achieving this result, for the reader's convenience, is recalled in appendix (B).

It is clear, however, that the assumption that s_{ik}, as defined by (1.2), can be the metric of Einstein-Schrödinger theory, is an hypothesis that needs further confirmation. Just the retrieval and the study of exact solutions of (A.2) - (A.5) displaying a wavy behaviour can either confirm or disprove the hypothesis [3]. Happily enough, two such solutions do exist. They belong to the class of exact solutions intrinsically depending on three coordinates [24], whose structure is recalled in appendix (C).

2. Wave propagation in two exact solutions

An exact solution allowing for wave propagation in two space and one time dimensions is easily built by the method of appendix (C), provided

[3]Needless to say, while a confirmation would be always provisional, a disproval would be a definitive one.

that one chooses

$$h_{ik} = \eta_{ik} \equiv \mathrm{diag}(-1,-1,-1,1) \tag{2.1}$$

as seed metric for the Hermitian solution, defined with respect to the coordinates $x^1 = x$, $x^2 = y$, $x^3 = z$, $x^4 = t$. In these coordinates the fundamental form g_{ik} of the mentioned solution reads:

$$g_{ik} = \begin{pmatrix} -1 & 0 & e & 0 \\ 0 & -1 & f & 0 \\ -e & -f & v & c \\ 0 & 0 & -c & 1 \end{pmatrix}, \tag{2.2}$$

with

$$v = -1 - c^2 + e^2 + f^2 \tag{2.3}$$

and

$$e = i\xi_{,x}, \ \ f = i\xi_{,y}, \ \ c = -i\xi_{,t}, \quad i = \sqrt{-1}, \tag{2.4}$$

where the function $\xi = \xi(x,y,t)$ fullfils, in the chosen representative space, the d'Alembert equation

$$\xi_{,xx} + \xi_{,yy} - \xi_{,tt} = 0 \tag{2.5}$$

with respect to the three coordinates x, y, t. When ξ is defined by (2.4), besides the field equation (A.3), also the unsolicited, invariant equation

$$g_{[ik],l} + g_{[kl],i} + g_{[li],k} = 0 \tag{2.6}$$

is satisfied, i.e. the antisymmetric field $g_{[ik]}$ appears to be endowed with electromagnetic meaning [4].

If the metric s_{ik}, defined by eq. (1.1), were equal to the seed metric h_{ik} defined by (2.1), the solution would represent electromagnetic waves that propagate with the fundamental velocity ($\mathrm{d}s^2 = 0$). The interval of the chosen metric s_{ik}, however, differs from the Minkowski interval. It is defined by equation (C.13) that, in the case of the particular solution defined by (2.2) - (2.4), reads

$$\mathrm{d}s^2 = s_{ik}\mathrm{d}x^i\mathrm{d}x^k = \sqrt{-v}(\mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2) + \frac{(\mathrm{d}\xi)^2}{\sqrt{-v}}. \tag{2.7}$$

If the second term at the right hand side of (2.7) were absent, the propagation of the electromagnetic waves would occur with the fundamental velocity also with respect to the chosen metric s_{ik}, because the first term at the right hand side is just conformally related to the square of the Minkowski interval. But a moment's reflection shows that, when ξ has, in the "Bildraum", a truly wave zone behaviour, hence a "Bildraum" wavevector can be locally defined, $\mathrm{d}\xi$, when taken along the direction of that wavevector, necessarily vanishes.

[4] Other solutions fulfilling (2.6), and representing the general electrostatic solution [25] and the magnetic field generated by constant electric currents running on n parallel wires [26] have been previously investigated by using s_{ik} as metric.

As a consequence $\mathrm{d}s^2$, as defined by (2.7), shall vanish in the direction of the "Bildraum" wavevector. One concludes that, with respect to the chosen metric s_{ik}, in the considered electromagnetic solution the electromagnetic waves in the wave zone do propagate with the fundamental velocity[5] in the metric sense ($\mathrm{d}s^2 = 0$).

Another exact solution, endowed with axial symmetry, and allowing too for wave propagation in two space and one time dimensions, is built by the same method of appendix (C), provided that one now chooses

$$h_{ik} = \mathrm{diag}(-1, -1, -r^2, 1), \tag{2.8}$$

defined with respect to polar cylindrical coordinates $x^1 = r$, $x^2 = z$, $x^3 = \varphi$, $x^4 = t$, as seed metric for the Hermitian solution. Its fundamental tensor g_{ik} reads:

$$g_{ik} = \begin{pmatrix} -1 & 0 & \delta & 0 \\ 0 & -1 & \varepsilon & 0 \\ -\delta & -\varepsilon & \zeta & \tau \\ 0 & 0 & -\tau & 1 \end{pmatrix}, \tag{2.9}$$

with

$$\zeta = -r^2 + \delta^2 + \varepsilon^2 - \tau^2, \tag{2.10}$$

and

$$\delta = ir^2\psi_{,r},\ \varepsilon = ir^2\psi_{,z},\ \tau = -ir^2\psi_{,t}, \tag{2.11}$$

where $\psi(r, z, t)$ now fullfils d'Alembert equation in cylindrical coordinates, namely:

$$\psi_{,rr} + \frac{\psi_{,r}}{r} + \psi_{,zz} - \psi_{,tt} = 0. \tag{2.12}$$

[5]The proof, that in the wave zone of the considered electromagnetic solution the electromagnetic waves do propagate with the fundamental velocity in the metric sense ($\mathrm{d}s^2 = 0$), is briefly outlined here, by availing of the very well known properties of D'Alembert's equation in the chosen "Bildraum". In a small neighbourhood of the wave zone, by suitable choice of the coordinates (otherwise, by suitable choice of the particular solution) equation (2.5) can be reduced to

$$\xi_{,xx} - \xi_{,tt} = 0.$$

In that small neighbourhood of the wave zone, a particular solution reads, say

$$\xi = \xi(x - t),$$

for which

$$\mathrm{d}\xi = 0$$

when

$$\mathrm{d}x = \mathrm{d}t,$$

as needed. Therefore, since $\mathrm{d}y = \mathrm{d}z = 0$, the interval $\mathrm{d}s^2$ defined by (2.7) vanishes locally in the wave zone.

The metric s_{ik} of this solution can be written as

$$s_{ik} = \frac{\sqrt{-\zeta}}{r}\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \frac{r^3}{\sqrt{-\zeta}}\begin{pmatrix} \psi_{,r}\psi_{,r} & \psi_{,r}\psi_{,z} & 0 & \psi_{,r}\psi_{,t} \\ \psi_{,r}\psi_{,z} & \psi_{,z}\psi_{,z} & 0 & \psi_{,z}\psi_{,t} \\ 0 & 0 & 0 & 0 \\ \psi_{,r}\psi_{,t} & \psi_{,z}\psi_{,t} & 0 & \psi_{,t}\psi_{,t} \end{pmatrix}, \tag{2.13}$$

hence the square of the line element, in the adopted coordinates, comes to read

$$\mathrm{d}s^2 = s_{ik}\mathrm{d}x^i\mathrm{d}x^k = \frac{\sqrt{-\zeta}}{r}\left(-\mathrm{d}r^2 - \mathrm{d}z^2 - r^2\mathrm{d}\varphi^2 + \mathrm{d}t^2\right) + \frac{r^3}{\sqrt{-\zeta}}(\mathrm{d}\psi)^2. \tag{2.14}$$

This solution has nothing to do with Maxwell's equations, because with the seed metric (2.8) the additional conditions (C.3) no longer have any relation to the electromagnetic looking equation (2.6).

A particular, time independent solution, obtained too from the same seed (2.8), for which

$$\psi = -\sum_{q=1}^{n} K_q \ln\frac{p_q + z - z_q}{r}, \tag{2.15}$$

where

$$p_q = [r^2 + (z - z_q)^2]^{1/2}, \tag{2.16}$$

while K_q and z_q are constants, has been investigated [27] some time ago. In keeping with an earlier approximate calculation done by Treder [19], it proves that pole sources at rest, defined by eq. (B.15), interact with forces not depending on their mutual distance, like the quarks of chromodynamics are supposed to do. The axially symmetric waves that we are presently considering should be therefore emitted and absorbed by such pole sources.

Whatever their physical meaning, the velocity with which these waves propagate is easily ascertained, like it occurred with the electromagnetic example considered previously. In fact, if the second term at the right hand side of (2.14) were absent, i.e. when $\mathrm{d}\psi$ is vanishing, the squared interval $\mathrm{d}s^2$, referred to cylindrical coordinates, would be conformally Minkowskian, and the speed of propagation of a wave with respect to the metric s_{ik} should be equal to the fundamental velocity that prevails with respect to the seed metric (2.8). But, again, a moment's reflection shows that, when ψ has, in the "Bildraum", a truly wave zone behaviour, hence when a "Bildraum" wavevector can be locally defined, $\mathrm{d}\psi$ necessarily vanishes, when taken along the direction of that wavevector. As a consequence $\mathrm{d}s^2$, as defined by (2.14), shall vanish in the direction of the "Bildraum" wavevector. Therefore, with the chosen metric s_{ik}, in the considered solution the waves of $g_{[ik]}$, whatever their physical meaning, do propagate in the wave zone with the fundamental velocity[6] in the metric sense ($\mathrm{d}s^2 = 0$).

In both the considered examples, the choice of the metric s_{ik} done by Kurşunoğlu [11, 12] and by Hély [14, 15] happens therefore to be compatible with the wavy behaviour exhibited by the exact solutions.

Appendix A. Einstein's unified field theory; Hermitian version

We consider here the Hermitian version [3] for Einstein's nonsymmetric generalization of the theory of 1915. A given geometric quantity [28] is called Hermitian with respect to the indices i and k, both either covariant or contravariant, if the part of the quantity that is symmetric with respect to i and k is real, while the part that is antisymmetric is purely imaginary. We contemplate the Hermitian fundamental form $g_{ik} = g_{(ik)} + g_{[ik]}$, and the affine connection $\Gamma^i_{kl} = \Gamma^i_{(kl)} + \Gamma^i_{[kl]}$, Hermitian with respect to the lower indices; both entities depend on the real coordinates x^i, while i runs from 1 to 4. We define also the Hermitian contravariant tensor g^{ik} through the relation

$$g^{il}g_{kl} \equiv g^{li}g_{lk} = \delta^i_k, \tag{A.1}$$

and the contravariant tensor density $\mathbf{g}^{ik} = (-g)^{1/2}g^{ik}$; $g \equiv \det(g_{ik})$ is a real quantity. Then the field equations of Einstein's unified field theory in the complex Hermitian form [3] come to read:

$$g_{ik,l} - g_{nk}\Gamma^n_{il} - g_{in}\Gamma^n_{lk} = 0, \tag{A.2}$$

$$\mathbf{g}^{[is]}_{\ \ ,s} = 0, \tag{A.3}$$

$$R_{(ik)}(\Gamma) = 0, \tag{A.4}$$

$$R_{[ik],l}(\Gamma) + R_{[kl],i}(\Gamma) + R_{[li],k}(\Gamma) = 0; \tag{A.5}$$

$R_{ik}(\Gamma)$ is the Hermitian Ricci tensor

$$R_{ik}(\Gamma) = \Gamma^a_{ik,a} - \Gamma^a_{ia,k} - \Gamma^a_{ib}\Gamma^b_{ak} + \Gamma^a_{ik}\Gamma^b_{ab}. \tag{A.6}$$

Appendix B. Adding phenomenological sources to all the Hermitian field equations of Einstein

In a four-dimensional manifold, let $\mathbf{g}^{ik}$ be a contravariant tensor density with an even part $\mathbf{g}^{(ik)}$ and an alternating one $\mathbf{g}^{[ik]}$:

$$\mathbf{g}^{ik} = \mathbf{g}^{(ik)} + \mathbf{g}^{[ik]}, \tag{B.1}$$

and W^i_{kl} be a general affine connection

$$W^i_{kl} = W^i_{(kl)} + W^i_{[kl]}. \tag{B.2}$$

For the Riemann curvature tensor built from W^i_{kl}:

$$R^i_{\ klm}(W) = W^i_{kl,m} - W^i_{km,l} - W^i_{al}W^a_{km} + W^i_{am}W^a_{kl}, \tag{B.3}$$

[6] A proof closely similar to the one given in the previous footnote applies here, and is left to the ingenuity of the reader.

two distinct contractions exist, $R_{ik}(W) = R^p{}_{ikp}(W)$ and $A_{ik}(W) = R^p{}_{pik}(W)$ [21]. But the transposed affine connection $\tilde{W}^i_{kl} = W^i_{lk}$ must be considered too: from it, the Riemann curvature tensor $R^i{}_{klm}(\tilde{W})$ and its two contractions $R_{ik}(\tilde{W})$ and $A_{ik}(\tilde{W})$ can be formed as well. We aim at following the pattern of the general relativity of 1915, which is built by the variational method from the Lagrangian density $\mathbf{g}^{ik}R_{ik}$, but now any linear combination $\bar{R}_{ik}$ of the four above-mentioned contractions is possible. A good choice [22], for physical reasons that will become apparent later, is

$$\bar{R}_{ik}(W) = R_{ik}(W) + \frac{1}{2}A_{ik}(\tilde{W}). \tag{B.4}$$

Let us provisionally endow the theory with sources in the form of a non-symmetric tensor P_{ik} and of a current density $\mathbf{j}^i$, coupled to $\mathbf{g}^{ik}$ and to the vector $W_i = W^l_{[il]}$ respectively. The Lagrangian density

$$\mathbf{L} = \mathbf{g}^{ik}\bar{R}_{ik}(W) - 8\pi\mathbf{g}^{ik}P_{ik} + \frac{8\pi}{3}W_i\mathbf{j}^i \tag{B.5}$$

is thus arrived at. By performing independent variations of the action $\int \mathbf{L}d\Omega$ with respect to W^p_{qr} and to $\mathbf{g}^{ik}$ with suitable boundary conditions we obtain the field equations

$$\begin{aligned} &-\mathbf{g}^{qr}{}_{,p} + \delta^r_p\mathbf{g}^{(sq)}{}_{,s} - \mathbf{g}^{sr}W^q_{sp} - \mathbf{g}^{qs}W^r_{ps} \\ &+\delta^r_p\mathbf{g}^{st}W^q_{st} + \mathbf{g}^{qr}W^t_{pt} = \frac{4\pi}{3}(\mathbf{j}^r\delta^q_p - \mathbf{j}^q\delta^r_p) \end{aligned} \tag{B.6}$$

and

$$\bar{R}_{ik}(W) = 8\pi P_{ik}. \tag{B.7}$$

By contracting eq. (B.6) with respect to q and p we get

$$\mathbf{g}^{[is]}{}_{,s} = 4\pi\mathbf{j}^i. \tag{B.8}$$

The very finding of this physically welcome equation entails however that we cannot determine the affine connection W^i_{kl} uniquely in terms of $\mathbf{g}^{ik}$: (B.6) is invariant under the projective transformation ${W'}^i_{kl} = W^i_{kl} + \delta^i_k\lambda_l$, where λ_l is an arbitrary vector field. Moreover eq. (B.7) is invariant under the transformation

$${W'}^i_{kl} = W^i_{kl} + \delta^i_k\mu_{,l} \tag{B.9}$$

where μ is an arbitrary scalar. By following Schrödinger [7, 21], we write

$$W^i_{kl} = \Gamma^i_{kl} - \frac{2}{3}\delta^i_k W_l, \tag{B.10}$$

where Γ^i_{kl} is another affine connection, by definition constrained to yield $\Gamma^l_{[il]=0}$. Then eq. (B.6) becomes

$$\mathbf{g}^{qr}{}_{,p} + \mathbf{g}^{sr}\Gamma^q_{sp} + \mathbf{g}^{qs}\Gamma^r_{ps} - \mathbf{g}^{qr}\Gamma^t_{(pt)} = \frac{4\pi}{3}(\mathbf{j}^q\delta^r_p - \mathbf{j}^r\delta^q_p) \tag{B.11}$$

that allows one to determine Γ^i_{kl} uniquely, under very general conditions [17, 20], in terms of $\mathbf{g}^{ik}$. When eq. (B.10) is substituted in eq. (B.7), the latter comes to read

$$\bar{R}_{(ik)}(\Gamma) = 8\pi P_{(ik)} \tag{B.12}$$

$$\bar{R}_{[ik]}(\Gamma) = 8\pi P_{[ik]} - \frac{1}{3}(W_{i,k} - W_{k,i}) \tag{B.13}$$

after splitting the even and the alternating parts. Wherever the source term is nonvanishing, a field equation loses its rôle, and becomes a definition of some property of matter in terms of geometrical entities; it is quite obvious that such a definition must be unique. This occurs with eqs. (B.8), (B.11) and (B.12), but it does not happen for eq. (B.13). This equation only prescribes that $\bar{R}_{[ik]}(\Gamma) - 8\pi P_{[ik]}$ is the curl of the arbitrary vector $W_i/3$; it is therefore equivalent to the four equations

$$\bar{R}_{[ik],l}(\Gamma) + \bar{R}_{[kl],i}(\Gamma) + \bar{R}_{[li],k}(\Gamma) = 8\pi\{P_{[ik],l} + P_{[kl],i} + P_{[li],k}\}, \tag{B.14}$$

that cannot specify $P_{[ik]}$ uniquely. We therefore scrap the redundant tensor $P_{[ik]}$, like we scrapped the redundant affine connection W^i_{kl} of eq. (B.6), and assume that matter is described by the symmetric tensor $P_{(ik)}$, by the conserved current density $\mathbf{j}^i$ and by the conserved current

$$K_{ikl} = \frac{1}{8\pi}\{\bar{R}_{[ik],l} + \bar{R}_{[kl],i} + \bar{R}_{[li],k}\}. \tag{B.15}$$

The general relativity of 1915, to which the present theory reduces when $\mathbf{g}^{[ik]} = 0$, suggests rewriting eq. (B.12) as

$$\bar{R}_{(ik)}(\Gamma) = 8\pi(T_{ik} - \frac{1}{2}s_{ik}s^{pq}T_{pq}) \tag{B.16}$$

where $s_{ik} = s_{ki}$ is the still unchosen metric tensor of the theory, $s^{il}s_{kl} = \delta^i_k$, and the symmetric tensor T_{ik} will act as energy tensor. If, in keeping with the choice done by Kurşunoğlu and by Hély, s_{ik} is defined like in equation (1.2), equation (B.16) is readily seen to stem directly from the variation of the Lagrangian (B.5), with a slightly reworked matter term, with respect to the chosen metric s_{ik}.

When sources are vanishing, equations (B.11), (B.16), (B.8) and (B.15) reduce to the original equations of Einstein's unified field theory, reported in appendix (A), because then $\bar{R}_{ik}(\Gamma)=R_{ik}(\Gamma)$; moreover they enjoy the property of transposition invariance also when the sources are nonvanishing. If $\mathbf{g}^{ik}$, Γ^i_{kl}, $\bar{R}_{ik}(\Gamma)$ represent a solution with the sources T_{ik}, $\mathbf{j}^i$ and K_{ikl}, the transposed quantities $\tilde{\mathbf{g}}^{ik} = \mathbf{g}^{ki}$, $\tilde{\Gamma}^i_{kl} = \Gamma^i_{lk}$ and $\bar{R}_{ik}(\tilde{\Gamma})= \bar{R}_{ki}(\Gamma)$ represent another solution, endowed with the sources $\tilde{T}_{ik} = T_{ik}, \tilde{\mathbf{j}}^i = -\mathbf{j}^i$ and $\tilde{K}_{ikl} = -K_{ikl}$. Such a physically desirable outcome is a consequence of the choice made [22] for $\bar{R}_{ik}$. These equations intimate that Einstein's unified field theory with sources should be interpreted like a gravoelectrodynamics in a polarizable continuum, allowing for both electric and magnetic currents. The study of the conservation identities confirms this idea [23] and strengthens at the same time the identification of the metric tensor s_{ik} done by Kurşunoğlu and by Hély. Let us consider the invariant integral

$$I = \int \left[\mathbf{g}^{ik}\bar{R}_{ik}(W) + \frac{8\pi}{3}W_i\mathbf{j}^i\right] d\Omega. \tag{B.17}$$

From it, when eq. (B.6) is assumed to hold, by means of an infinitesimal coordinate transformation we get the four identities

$$\begin{aligned} -(\mathbf{g}^{is}\bar{R}_{ik}(W) + \mathbf{g}^{si}\bar{R}_{ki}(W))_{,s} + \mathbf{g}^{pq}\bar{R}_{pq,k}(W) \\ + \frac{8\pi}{3}\mathbf{j}^i(W_{i,k} - W_{k,i}) = 0. \end{aligned} \tag{B.18}$$

This equation can be rewritten as

$$\begin{aligned} -2(\mathbf{g}^{(is)}\bar{R}_{(ik)}(\Gamma))_{,s} + \mathbf{g}^{(pq)}\bar{R}_{(pq),k}(\Gamma) \\ = 2\mathbf{g}^{[is]}_{\ ,s}\bar{R}_{[ik]}(\Gamma) + \mathbf{g}^{[is]}\left\{\bar{R}_{[ik],s}(\Gamma) + \bar{R}_{[ks],i}(\Gamma) + \bar{R}_{[si],k}(\Gamma)\right\} \end{aligned} \tag{B.19}$$

where the redundant variable W^i_{kl} no longer appears. The metric tensor s_{ik} is defined by equation (1.2), and just for the tensor T_{ik} we shall make an exception to the general rule that prevails in the Einstein-Schrödinger theory, and use s^{ik} and s_{ik} to raise and lower indices, $\sqrt{-s}$ to produce tensor densities out of tensors. We define then

$$\mathbf{T}^{ik} = \sqrt{-s}s^{ip}s^{kq}T_{pq} \tag{B.20}$$

and the weak identities (B.19), when all the field equations hold, will take the form

$$\mathbf{T}^{ls}_{;s} = \frac{1}{2}s^{lk}(\mathbf{j}^i\bar{R}_{[ki]}(\Gamma) + K_{iks}\mathbf{g}^{[si]}), \tag{B.21}$$

where the semicolon means covariant derivative with respect to the Christoffel affine connection

$$\left\{ {i \atop k\ l} \right\} = \frac{1}{2}s^{im}(s_{mk,l} + s_{ml,k} - s_{kl,m}) \tag{B.22}$$

built with s_{ik}. As far as one knows, only the choice (1.2) of the metric and the way of appending sources adopted in eqs. (B.11), (B.8), (B.15) and (B.16) allows rewriting the identities (B.19) in so simple and so physically suggestive a way. The previous impression is strengthened by eq. (B.21): the theory, built in terms of a non-Riemannian geometry, appears to entail a gravoelectrodynamics in a dynamically polarized Riemannian spacetime, for which s_{ik} is the metric, where the two conserved currents $\mathbf{j}^i$ and K_{iks} are coupled à la Lorentz to $\bar{R}_{[ki]}$ and to $\mathbf{g}^{[si]}$ respectively. Two versions of this gravoelectrodynamics are possible, according to whether $\mathbf{g}^{ik}$ is chosen to be either a real nonsymmetric or a complex Hermitian tensor density, like we presently do. The constitutive relation between electromagnetic inductions and fields is governed by the field equations in a quite novel and subtle way: the link between $\mathbf{g}^{[ik]}$ and $\bar{R}_{[ik]}$ is not the simple algebraic one usually attributed to the vacuum, with some metric that raises or lowers indices, and builds densities from tensors. It is a differential one, and a glance to the field equations suffices to become convinced that understanding its properties is impossible without first finding and perusing the exact solutions of the theory.

Appendix C. Solutions of the Hermitian theory that depend on three coordinates

We assume that Greek indices take the values 1,2 and 4, while Latin indices run from 1 to 4. Let the real symmetric tensor h_{ik} be the metric for a vacuum solution to the field equations of the general relativity of 1915, which depends on the three co-ordinates x^λ, not necessarily all spatial in character, and for which $h_{\lambda 3} = 0$. We consider also an antisymmetric purely imaginary tensor a_{ik}, which depends too only on the co-ordinates x^λ, and we assume that its only nonvanishing components are $a_{\mu 3} = -a_{3\mu}$. Then we form the mixed tensor

$$\alpha_i{}^k = a_{il} h^{kl} = -\alpha^k{}_i, \tag{C.1}$$

where h^{ik} is the inverse of h_{ik}, and we define the Hermitian fundamental form g_{ik} as follows:

$$\begin{aligned} g_{\lambda\mu} &= h_{\lambda\mu}, \\ g_{3\mu} &= \alpha_3{}^\nu h_{\mu\nu}, \\ g_{33} &= h_{33} - \alpha_3{}^\mu \alpha_3{}^\nu h_{\mu\nu}. \end{aligned} \tag{C.2}$$

When the three additional conditions

$$\alpha^3{}_{\mu,\lambda} - \alpha^3{}_{\lambda,\mu} = 0 \tag{C.3}$$

are fulfilled, the affine connection Γ^i_{kl} which solves eqs. (A.2) has the non-vanishing components

$$\begin{aligned} \Gamma^\lambda_{(\mu\nu)} &= \left\{ {\lambda \atop \mu\ \nu} \right\}_{(h)}, \\ \Gamma^\lambda_{[3\nu]} &= \alpha_3{}^\lambda{}_{,\nu} - \left\{ {3 \atop 3\ \nu} \right\}_{(h)} \alpha_3{}^\lambda + \left\{ {\lambda \atop \rho\ \nu} \right\}_{(h)} \alpha_3{}^\rho, \\ \Gamma^3_{(3\nu)} &= \left\{ {3 \atop 3\ \nu} \right\}_{(h)}, \\ \Gamma^\lambda_{33} &= \left\{ {\lambda \atop 3\ 3} \right\}_{(h)} - \alpha_3{}^\nu \left(\Gamma^\lambda_{[3\nu]} - \alpha_3{}^\lambda \Gamma^3_{(3\nu)} \right); \end{aligned} \tag{C.4}$$

we indicate with $\left\{ {i \atop k\ l} \right\}_{(h)}$ the Christoffel connection built with h_{ik}. We form now the Ricci tensor (A.6). When eqs. (A.3), i.e., in our case, the single equation

$$(\sqrt{-h}\ \alpha_3{}^\lambda h^{33})_{,\lambda} = 0, \tag{C.5}$$

and the additional conditions, expressed by eqs. (C.3), are satisfied, the components of $R_{ik}(\Gamma)$ can be written as

$$R_{\lambda\mu} = H_{\lambda\mu},$$
$$R_{3\mu} = \alpha_3{}^{\nu} H_{\mu\nu} + \left(\alpha_3{}^{\nu} \left\{ {}^{\,3}_{3\ \nu} \right\}_{(h)}\right)_{,\mu}, \tag{C.6}$$
$$R_{33} = H_{33} - \alpha_3{}^{\mu}\alpha_3{}^{\nu} H_{\mu\nu},$$

where H_{ik} is the Ricci tensor built with $\left\{ {}^{\ i}_{k\ l} \right\}_{(h)}$. H_{ik} is zero when the seed metric h_{ik} is a vacuum solution of the field equations of general relativity, as supposed; therefore, when eqs. (C.3) and (C.5) hold, the Ricci tensor, defined by eqs. (C.6), satisfies eqs. (A.4) and (A.5) of the Hermitian theory of relativity.

The task of solving equations (A.2)-(A.5) reduces, under the circumstances considered here, to the simpler task of solving eqs. (C.3) and (C.5) for a given h_{ik}.[7]

Let us suppose that the metric tensor is defined by the equation (1.2), namely

$$\sqrt{-s}\, s^{ik} = \mathbf{g}^{(ik)}, \tag{C.7}$$

where $s^{il}s_{kl} = \delta^i_k$ and $s = \det(s_{ik})$. When the fundamental tensor g_{ik} has the form (C.2) it is

$$\sqrt{-g} = \sqrt{-h}, \tag{C.8}$$

where $h \equiv \det(h_{ik})$, and

$$\det\left(g^{(ik)}\right) = \frac{1 - g^{3\tau}g_{3\tau}}{h}. \tag{C.9}$$

Therefore

$$\sqrt{-s} = \sqrt{-h}\left(1 - g^{3\tau}g_{3\tau}\right)^{1/2}, \tag{C.10}$$

hence

$$s^{ik} = g^{(ik)}\left(1 - g^{3\tau}g_{3\tau}\right)^{-1/2}. \tag{C.11}$$

The nonvanishing components of s_{ik} then read

$$s_{\lambda\mu} = \left(1 - g^{3\tau}g_{3\tau}\right)^{1/2} h_{\lambda\mu} + \left(1 - g^{3\tau}g_{3\tau}\right)^{-1/2} h_{33}\alpha^3{}_{\lambda}\alpha^3{}_{\mu},$$
$$s_{33} = \left(1 - g^{3\tau}g_{3\tau}\right)^{1/2} h_{33}, \tag{C.12}$$

and the square of the interval $\mathrm{d}s^2 = s_{ik}\mathrm{d}x^i\mathrm{d}x^k$ eventually comes to read

$$\mathrm{d}s^2 = \left(1 - g^{3\tau}g_{3\tau}\right)^{1/2} h_{ik}\mathrm{d}x^i\mathrm{d}x^k - \left(1 - g^{3\tau}g_{3\tau}\right)^{-1/2} h_{33}\left(\mathrm{d}\xi\right)^2. \tag{C.13}$$

In keeping with (C.3), we have defined $\alpha^3{}_{\mu}$ as

$$\alpha^3{}_{\mu} = i\xi_{,\mu}, \tag{C.14}$$

in terms of the real function $\xi(x^\lambda)$.

[7] This method of solution obviously applies to Schrödinger's purely affine theory [7] too.

References

1. Einstein, A. (1925). *S. B. Preuss. Akad. Wiss.*, **22**, 414.
2. Einstein, A., and Straus, E.G. (1946). *Ann. Math.*, **47**, 731.
3. Einstein, A. (1948). *Rev. Mod. Phys.*, **20**, 35.
4. Einstein, A., and Kaufman, B. (1955). *Ann. Math.*, **62**, 128.
5. Schrödinger, E. (1947). *Proc. R. I. Acad.*, **51A**, 163.
6. Schrödinger, E. (1947). *Proc. R. I. Acad.*, **51A**, 205.
7. Schrödinger, E. (1948). *Proc. R. I. Acad.*, **52A**, 1.
8. Schrödinger, E. (1951). *Proc. R. I. Acad.*, **54A**, 79.
9. Papapetrou, A. (1948). *Proc. R. I. Acad.*, **52A**, 69.
10. Wyman, M. (1950). *Can. J. Math.*, **2**, 427.
11. Kurşunoğlu, B. (1952). *Proc. Phys. Soc. A*, **65**, 81.
12. Kurşunoğlu, B. (1952). *Phys. Rev.*, **88**, 1369.
13. Lichnerowicz, A., (1954). *J. Rat. Mech. Anal.*, **3**, 487.
14. Hély, J. (1954). *Comptes Rend. Acad. Sci. (Paris)*, **239**, 385.
15. Hély, J. (1954). *Comptes Rend. Acad. Sci. (Paris)*, **239**, 747.
16. Lichnerowicz, A., (1955). *Théories relativistes de la gravitation et de l'électromagnétisme*, Masson, Paris.
17. Tonnelat, M. A. (1955). *La Théorie du Champ Unifié d'Einstein*, Gauthier-Villars, Paris.
18. Narlikar, V.V., and Rao, B.R. (1956). *Proc. Nat. Inst. Sci. India* **21A**, 409.
19. Treder, H. (1957). *Ann. Phys. (Leipzig)*, **19**, 369.
20. Hlavatý, V. (1957). *Geometry of Einstein's Unified Field Theory* Noordhoff, Groningen.
21. Schrödinger, E. (1954). *Space-Time Structure*, Cambridge University Press, Cambridge.
22. Borchsenius, K. (1978). *Nuovo Cimento*, **46A**, 403.
23. Antoci, S. (1991). *Gen. Rel. Grav.*, **23**, 47. Also: http://arxiv.org/abs/gr-qc/0108052.
24. Antoci, S. (1987). *Ann. Phys. (Leipzig)*, **44**, 297; http://arXiv.org/abs/gr-qc/0108042.
25. Antoci, S., Liebscher, D.-E. and Mihich, L. (2005). *Gen. Rel. Grav.*, **37**, 1191; http://arxiv.org/abs/gr-qc/0405064.
26. Antoci, S. (2009). *Nuovo Cimento* **124B**, 121; http://arxiv.org/abs/0803.3587.
27. Antoci, S., Liebscher, D.-E. and Mihich, L. (2008). *Annales Fond. de Broglie*, **33**, 221; http://arxiv.org/abs/gr-qc/0604003.
28. Schouten, J.A. (1954). *Ricci-calculus; an introduction to tensor analysis and its geometrical applications*, Springer, Berlin.

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