

THE GROUP ASPECT IN THE PHYSICAL INTERPRETATION OF GENERAL RELATIVITY THEORY

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When, at the end of the year 1915, both Einstein and Hilbert arrived at what were named the field equations of general relativity, both of them thought that their fundamental achievement entailed, inter alia, the realisation of a theory of gravitation whose underlying group was the group of general coordinate transformations. This group theoretical property was believed by Einstein to be a relevant one from a physical standpoint, because the general coordinates allowed to introduce reference frames not limited to the inertial reference frames that can be associated with the Minkowski coordinate systems, whose transformation group was perceived to be restricted to the Poincaré group.

Two years later, however, Kretschmann published a paper in which the physical relevance of the group theoretical achievement in the general relativity of 1915 was denied. For Kretschmann, since any theory, whatever its physical content, can be rewritten in a generally covariant form, the group of general coordinate transformations is physically irrelevant. This is not the case, however, for the group of the infinitesimal motions that bring the metric field in itself, namely, for the Killing group. This group is physically characteristic of any given spacetime theory, since it accounts for the local invariance properties of the considered manifold, i.e., for its “relativity postulate”.

In Kretschmann’s view, the so called restricted relativity of 1905 is the one with the relativity postulate of largest content, because the associated Killing group coincides with the infinitesimal Poincaré group, while for the most general metric manifold of general relativity the associated Killing group happens to contain only the identity, hence the content of its relativity postulate is nil.

Of course, solutions to the field equations of general relativity whose relativity postulate has a content that is intermediate between the two above mentioned extremes exist too. They are the ones generally found and investigated until now by the relativists, since the a priori assumption of some nontrivial Killing invariance group generally eases the finding of solutions to the above mentioned equations. In the present chapter it is shown what are the consequences for the physical interpretation of some of these solutions whose relativity postulate is of intermediate content, when Kretschmann’s standpoint is consistently adhered to.

1. INTRODUCTION

It may seem strange that, in the year 2009, one may conceive writing a text with the title given above. So many years have elapsed since Einstein [1] and Hilbert [2] eventually wrote the final equations of general relativity theory, and it might be reasonable to believe that by now the issue hinted at in the title should have been settled once and for all. The rôle played by group theory in the so called general theory of relativity should be clear beyond discussion, and no further paper should need to be written on this subject.

However, this is not the case. Moreover, this problem has emerged as early as in the year 1917, when Erich Kretschmann defied the group theoretical assessment given by Einstein [3], and proposed an alternative of his own [4], whose validity in principle was to be soon acknowledged by Einstein himself [5]. In Frank's review [6] of Kretschmann's paper one finds a short, but precise account of the main points considered by Kretschmann. It reads, in English translation:

“Einstein understands, under his general principle of relativity, the injunction that the laws of nature must be expressed through equations that are covariant with respect to arbitrary coordinate transformations. The Author shows now that any natural phenomenon obeying any law can be described by generally covariant equations. Therefore the existence of such equations does not express any physical property. For instance the uniform propagation of light in a space free from gravitation can be expressed also in a covariant way. However, there is a representation of the same phenomena, that admits only a more restricted group (the Lorentz transformations). This group, that cannot be further restricted by any representation of the phenomena, is characteristic of the system under question. The invariance with respect to it is a physical property of the system and, in the sense of the Author, it represents the postulate of relativity for the corresponding domain of phenomena.

In Einstein's general theory of relativity, through appropriate choice of the coordinates, the field equations can be converted in a form that is no longer covariant under the group of coordinate transformations. The Author provides a series of examples of such conversions. But the equations converted in this way in general no longer admit any group, and in this sense Einstein's theory of general relativity is an “absolute theory”, while the special theory of relativity satisfies the postulate of relativity for the Lorentz transformations also in the sense of the Author.”

When reading Kretschmann's paper today, one confronts its lengthly, sometimes obscure pages with a growing sense of admiration for the keen physical intuition that drove its author to a right conclusion despite his lack of the correct mathematical tools for tackling the difficult questions that he addressed, and forced him to try, one after another, several paths of thought that he critically evaluated not to be fully satisfactory in one way or another.

In the present chapter Kretschmann's comparison between the group theoretical assessment of the special and of the general theory of relativity is reconsidered by availing of the mathematical tool that is lacking in Kretschmann's work, i.e. the group of infinitesimal Killing motions to be associated to each theory endowed with a metric tensor. If the group properties of both flat and curved spaces need to be compared through the same mathematical tool, it is this group that must take the rôle of what Kretschmann calls the group of invariance, the one endowed with physical meaning, that he so many times invokes in his paper as the one needed for properly assessing the "relativity postulate" of each theory. From this recognition several relevant consequences immediately follow for the way that must be kept when physically interpreting the solutions to the field equations of general relativity endowed with nontrivial groups of invariance.

But let us go back at present to special relativity and to its group theoretical assessment, that has constituted the mathematically and physically sound paradigm from which Kretschmann has moved for building his interpretation of general relativity as an "absolute theory".

2. FINDING THE GROUP OF INVARIANCE IN SPECIAL AND IN GENERAL RELATIVITY

The reader shall forgive us if we recall here concepts that have been perused in all the textbooks of relativity since a long time. We need to do so for retrieving a mathematical formulation, that may have the distinct advantage of being applicable without change both to the special and to the general theory of relativity.

The Poincaré group of transformations between the inertial coordinate frames ¹ of special relativity can be given in principle many representations. The generally adopted one relies on what Landau and Lifshits [7] once called "Galilean coordinates" $x^1 = x$, $x^2 = y$, $x^3 = z$, $x^4 = t$, and on the Minkowski metric, expressed with respect to these coordinates:

$$(2.1) \quad \eta_{ik} = \text{diag}(-1, -1, -1, 1),$$

that is invariant ² under the coordinate transformations of the Poincaré group. When this representation is adopted, the coordinates are not just labels for identifying events; due to the particular form of η_{ik} , they have a direct metric reading, i.e. to each particular system of coordinates a physically admissible reference frame, to be built with rods, clocks and light signals, is directly associated in one-to-one correspondence. As a consequence, by

¹The notion reference frame acquired in the meantime a meaning different from coordinate system, although in special relativity both are usually intimately connected.

²One cannot help recalling here the ironic sentence by Felix Klein: "Was die modernen Physiker *Relativitätstheorie* nennen, ist die Invariantentheorie des vierdimensionalen Raum-Zeit-Gebietes, x, y, z, t (der Minkowskischen "Welt") gegenüber einer bestimmten Gruppe von Kollineationen, eben der "Lorentzgruppe" [8].

availing of this representation, one recognizes that the Poincaré group, besides being, from a mathematical standpoint, the group of invariance of η_{ik} , is endowed with direct physical meaning. The invariance of η_{ik} under the Poincaré group constitutes what Kretschmann once called the physically meaningful “relativity postulate” of the original theory of relativity.

However, it is quite possible, and Kretschmann was fully aware of this [9, 4], that one accounts for special relativity by adopting a general system of curvilinear coordinates, with the associated group of general coordinate transformations. This move has the distinct advantage of freeing the coordinate systems from the duplicity of function that they play in the previous account, i.e. both providers of labels for the identification of the events, and elements to which the transformations of the invariance group directly apply. We do not know whether these coordinate systems can maintain a physical rôle beyond the purely topological one of identifying the events, namely, whether reference frames can be associated with these curvilinear coordinates too, as it was hoped for by Einstein [3]; today, the answer to the above question is the identification of a reference frame at a given event with the vector base in its tangent Minkowski space-time ³. However, we are sure that in general relativity the latter coordinate systems have no relation whatsoever with the physically relevant invariance group of the special relativity theory ⁴.

The adoption of curvilinear coordinates for expressing the theory of special relativity is fundamental for acknowledging that the restriction of the allowed coordinate systems to the ones corresponding to the inertial frames, although very intuitive and convenient for the calculations, is conceptually inessential. The eventual recognition of the group of invariance of the metric in a given theory is the true scope that we aim at, either in the special or in the general theory of relativity, and we shall equip ourselves with the appropriate mathematical tool. Since the absolute differential calculus of Ricci and Levi Civita is naturally expressed with curvilinear coordinates, these shall constitute an appropriate choice for accomplishing our task.

There is a fundamental difference between the special and the general theory of relativity, that is decisive for the very choice of the group of invariance that we shall look at in both cases, and for the unique mathematical tool that we shall eventually adopt for the comparison. In special relativity, as it is evident just because Galilean coordinates can be used in the double rôle explained above, the representation of the group of invariance has a global character, while in a nontrivial pseudo Riemannian manifold a group

³In this way, one obtains a field of frames which generates a teleparallel transport with torsion instead of curvature. The transformations of the frames form the Lorentz group at each point separately, and event-dependent Lorentz transformations for the field of frames. Eventually, already Einstein tried to generalize the theory to implement the electromagnetic field in this direction.

⁴The choice of the coordinate system as well as that of the field of frames do not enter any observable here.

of invariance of the metric, if it exists at all, in general can be identified mathematically only in the infinitesimal neighbourhood of each event. The several, keen but unsuccessful attempts by Kretschmann to provide a global identification of the invariance group through explicit analytic or geometric procedures both in the case of special relativity as seen in curvilinear coordinates, and in the case of general relativity, testify the difficulty of the global problem, on which scarce progress has occurred since Kretschmann's times.

Happily enough, if we investigate the invariance group of the metric in the infinitesimal neighbourhood of each event, by availing of the powerful tools provided by Lie and by Killing [10] we can identify and use the algebra of the Killing vectors that prevails in each one of these neighbourhoods, both in the special and in the general theory of relativity. The conceptual problem is thereby reduced to the mathematical problem of finding the solutions of the Killing equations (A.5) of Appendix A and of studying the group properties of the infinitesimal Killing motions found in this way. As it is evident from Appendix A, the group of the infinitesimal Killing motions does not deal with infinitesimal point transformations: by its very nature, this method analyses the invariance group of the metric under infinitesimal "Mitschleppen" (dragging along). This change of objective may appear inessential for special relativity, due to the homogeneous character of the considered manifold. In this case, the global answer that the invariance group of the metric is the Poincaré group can be reached anyway, by starting from the infinitesimal Killing group, only through a more complicated argument. The study of the invariance group for infinitesimal "Mitschleppen", however, is the only one that is possible in general for a pseudo Riemannian, curved manifold. The infinitesimal Killing vector group is therefore the tool for realizing Kretschmann's program of comparison of the invariance groups of the metric that prevail in the special and in the general theory of relativity respectively.

The search for the Killing group for both special and general relativity is straightforward and confirms Kretschmann's objection of 1917: while the Killing group of the metric of special relativity is the Poincaré group for infinitesimal motions, for a general solution of the field equations of general relativity the Killing group reduces to the identity, i.e. general relativity, despite its very name, is indeed an absolute theory.

3. APPLYING KRETSCHMANN STANDPOINT TO SOLUTIONS WITH INTERMEDIATE RELATIVITY POSTULATE

Finding exact solutions to the field equations of general relativity is a very demanding task; no wonder then if in the decades-long search for new solutions, since when Karl Schwarzschild discovered the spherically symmetric, static solution that bears his name [13], the problem has been generally eased by limiting the search to the simpler solutions for which the Killing groups

of the metric are intermediate between the one of special relativity and the one, endowed only with the identity, of the most general solutions of general relativity. As a consequence, the invariance groups of the metric fields that we can really explore are nontrivial and, according to Kretschmann's standpoint, intrinsic physical content is introduced a priori. Let us notice that the idea of a particular physical content associated with a particular nontrivial invariance group of the metric is fully in keeping with the findings by Hilbert, Klein and with the fundamental result by Noether [2, 11, 12] about the essential link between invariance and conservation laws.

We are therefore confronted with a very interesting, but really difficult situation. The very fact that in general relativity each particular solution of the field equations exhibits its own particular content of the physically relevant invariance group is a novel feature that counters our expectations. We were prepared to search for a unique, once for all theory of the observables of general relativity, like it happens in special relativity, for which the Killing group is fixed from the outset. In general relativity these observables should behave as scalars under the group of coordinate transformations, because tensor quantities depend on the choice of the coordinates, which are today generally presumed to be mere labels for identifying events, otherwise devoid of physical meaning⁵. But we do not know how to find general exact solutions, for which these observables might display their full structure and meaning, and even if we could find these solutions and calculate their observables, the latter could not have any resemblance to the observables of special relativity. In fact, besides being invariant quantities, we know in advance that they would obey no genuine conservation law, since the Killing group of such general solutions would contain only the identity. We must content ourselves, however, with the examples provided by the particular solutions endowed with a nontrivial Killing group which, if the Riemann tensor is nonvanishing, is different from and endowed with less elements than the Poincaré group.

Let us explore, by availing of Kretschmann's and Noether's standpoint, some well known solutions of general relativity, like the Schwarzschild solution, both with the original, pondered choice of the manifold done by Schwarzschild [13] himself and in the form, endowed with an inequivalent manifold, accidentally introduced⁶ by Hilbert [14], as well as its Kruskal-Szekeres maximal extension [17, 18]; the Kerr-Newman solution [19, 20] will

⁵Scalars obtained by considering the tetrad components of some tensor with respect to some tetrad field would be equally devoid of physical meaning, due to the arbitrariness in the choice of the tetrad field.

⁶For a historical account on Schwarzschild's original manifold and on the inequivalent choice of the manifold done by Hilbert, one may consult [15] and [16].

be considered too. There is also a body of literature on the so-called boost-rotation symmetric solutions⁷ that seems worth of analysis. The perusal of these manifolds from the above mentioned standpoint leads to disconcerting results. All these solutions, with the exception of Schwarzschild's original manifold [13], have one feature in common: the manifold, on which the solution is defined, happens to be built from the juxtaposition of submanifolds endowed with different invariance groups of the metric, hence with different intrinsic physical meaning, because, according to Noether [12], the quantities that are conserved in each one of the submanifolds are physically different.

This peculiar behaviour, common to the solutions mentioned above, with the exception of Schwarzschild's original solution, is invariably due to the presence, within the manifold, of surfaces on which the character of one Killing vector field changes from timelike to spacelike or vice versa, with a consequent change of the physical meaning of the prevailing Killing group when one crosses one such surface of junction between neighbouring submanifolds. This is a well known behaviour, but the danger of allowing in this way for intrinsically nonsensical, patchwork manifolds, with unrelated physical processes, subject to unrelated conservation laws going on severally in each of the submanifolds, has been intimated only recently [33].

The adoption of such composite manifolds as models of some physical reality has occurred because the criteria adopted for their selection have been based exclusively on the two very important notions of local singularity and of geodesic completeness. The two notions are deeply intertwined in the studies that have been developed during many years while searching for a general, invariant and physically satisfactory definition of singular boundary in general relativity⁸. It is not here the place for recalling them in extenso. Suffice it to say that the notion of intrinsic, local singularity has been associated with the divergent behaviour of the polynomial invariants built with the metric g_{ik} , with the Levi-Civita symbol ϵ^{iklm} , with the Riemann tensor R_{iklm} and with its covariant derivatives, when some limit boundary is approached along a geodesic path. A manifold is said to be geodesically complete when its geodesics either can be defined for any value of their affine parameter, or meet some limit boundary where some of the above mentioned polynomial invariants diverge. The occurrence of the latter divergence is of course an appropriate, sufficient condition for defining a singularity intrinsic to the manifold, and the requirement of geodesic completeness is likely to be a geometrically and physically correct regularity criterion for a general solution, for which the Killing group reduces to the identity.

⁷From the references on the subject let us quote here only the solutions with non-spinning sources, reported and investigated in [23], [24], [25], [26], [27, 28, 29], [30, 31], [32].

⁸see for instance [34, 35, 36, 37] [38, 39, 40, 41].

When this criterion is applied to the solutions mentioned above, for which the Killing group does not reduce to the identity, the following assessment is reached:

- Schwarzschild’s original manifold [13] is defective, because of geodesic incompleteness. No geodesic reaching an intrinsic singularity due to the divergence of some polynomial invariant of the Riemann tensor can be drawn on it.
- Hilbert’s manifold is defective due to geodesic incompleteness too. Geodesics hitting an intrinsic singularity of the previously defined kind, or emanating from it, can be drawn, but one fails to assign to them a proper arrow of time ⁹.
- The Kruskal-Szekeres manifold [17, 18] is geodesically complete and has a proper arrow of time.
- The Kerr-Newman manifold [19, 20] lacks geodesic completion and does not have a proper arrow of time. Both the Kerr and the Reissner - Nordström [21, 22] manifolds have been severally completed.
- The so-called boost-rotation symmetric manifolds of [23]-[32] generally await geodesic completion.

When confronted with the diagram of the Kruskal-Szekeres manifold, the perception that a consequent reasoning has eventually led us to acknowledge the need of these four quadrants for properly describing, in general relativity, the gravitational field of one material particle at rest, this perception has been sufficient to raise in some relativists the following doubt. The faultless logic of the program of geodesic completion is of course likely to be quite correct for a general solution to the field equations of general relativity, for which the invariance group of the metric contains only the identity, hence it is irrelevant. Is not it possible that the same program may instead lead us astray when applied to manifolds that happen to be invariantly, intrinsically divided in submanifolds, because nontrivial and physically different invariance groups of the metric prevail in different parts of the complete manifold?

The further consideration of the infinite repetitions occurring in the diagrams needed to perfect the program of geodesic completion for both the Kerr and the Reissner-Nordström solutions cannot but strengthen the doubt raised already by the Kruskal-Szekeres manifold, and leads one to wonder whether something similar to what occurred to Goethe’s “Zauberlehrling” is happening here.

⁹As required by Synge [42], a proper arrow of time shall satisfy both the postulate of *order*, according to which the affine parameter on one geodesic is always increasing or decreasing when one goes along the geodesic in a given sense, and the *non-circuital* postulate, according to which one cannot build, with segments of geodesics, a closed loop on which the time arrow always points in the same sense. For the arrow of time in Hilbert’s manifold see also [43, 16].

If one imposes instead the condition that, in order to be a model of some physical reality, a manifold must not contain in its interior local, invariant, intrinsic singularities, and must be endowed with a unique group of invariance, the assessment of the solutions previously considered becomes the following:

- Schwarzschild's original manifold fulfills the condition.
- Hilbert's manifold, that we consider here in the usual coordinate system due to Hilbert [14], does not fulfill the condition because the hypersurface orthogonal, timelike Killing vector that can be uniquely drawn at each event for which $r > 2m$ becomes spacelike for $0 < r < 2m$.
- The Kruskal-Szekeres manifold does not fulfill the condition for the same reason as the one prevailing with Hilbert's manifold.
- The interval of the Kerr-Newman manifold, expressed in Boyer-Lindquist coordinates, reads:

$$\begin{aligned}
 (3.1) \quad ds^2 &= -\frac{\varrho^2}{\Delta} dr^2 - \varrho^2 d\vartheta^2 - \frac{\sin^2 \vartheta}{\varrho^2} ((r^2 + J^2) d\varphi - J dt)^2 \\
 &+ \frac{\Delta}{\varrho^2} (dt - J \sin^2 \vartheta d\varphi)^2, \\
 \Delta &= r^2 + J^2 + Q^2 - 2Mr, \\
 \varrho^2 &= r^2 + J^2 \cos^2 \vartheta,
 \end{aligned}$$

and the manifold does not fulfill the condition. A uniform Killing group structure prevails for $r > r_0 = M + \sqrt{M^2 - J^2 - Q^2}$. One can fulfill the condition by ending the manifold there.

- The boost-rotation symmetric manifolds quoted in footnote [7] do not fulfill the condition, because they are obtained through the juxtaposition of submanifolds endowed with physically different groups of invariance. Again, we are confronted with a hypersurface orthogonal, timelike Killing vector that becomes spacelike on crossing certain hypersurfaces [33].

4. THE SINGULAR BORDER BETWEEN SUBMANIFOLDS ENDOWED WITH DIFFERENT INVARIANCE GROUPS

Despite the fact that the submanifolds into which the previously considered solutions have been divided are invariantly defined, one might still wonder why one should truncate manifolds that are geodesically complete, when no singularities defined through the polynomial invariants of the Riemann tensor occur at the borders produced in that way, and when regular geodesics can be drawn across them. The question can be answered by remarking that geodesics are very special worldlines, and that the regularity of all the worldlines either crossing such borders or lying closer and closer to them should be investigated.

With the manifolds considered in the previous section, however, one does not need to accomplish such a cumbersome program for reaching the answer. The nontrivial Killing structure of the considered solutions allows in fact the definition of local, invariant, intrinsic quantities besides the just mentioned polynomial invariants, and these quantities happen to exhibit a divergent, singular behaviour when the borders between submanifolds endowed with different invariance groups are approached.

The Killing group of Schwarzschild's original manifold [13], hence the Killing structure of both the submanifold of Hilbert's solution [14] for $r > 2m$, and of the left and right quadrants of the Kruskal-Szekeres manifold [17, 18], define at each event a unique [44], hypersurface-orthogonal, timelike Killing vector ξ_i :

$$(4.1) \quad \xi_i \xi^i > 0, \quad \xi_{i;k} + \xi_{k;i} = 0, \quad \xi_{[i} \xi_{k,l]} = 0.$$

Due to its uniqueness, and since each hypersurface orthogonal to it is space-like, this vector defines the unique direction of *absolute rest* in the manifold where it prevails, and allows to build congruences of absolute rest. Let us calculate the first curvature of one such congruence, i.e. the four-acceleration

$$(4.2) \quad a^i = \frac{D u^i}{ds} \equiv \frac{d u^i}{ds} + \Gamma_{kl}^i u^k u^l,$$

where D/ds indicates the absolute derivative, and $u^i = dx^i/ds$ is the four-velocity tangent to the chosen congruence. From it one builds the norm

$$(4.3) \quad \alpha = (-a_i a^i)^{1/2}.$$

Due to its very definition, this local, invariant quantity is also intrinsic to the manifold where it prevails. When Schwarzschild's solution is written by using Hilbert's coordinates $x^1 = r$, $x^2 = \vartheta$, $x^3 = \phi$, $x^4 = t$, its interval reads

$$(4.4) \quad ds^2 = (1 - 2m/r) dt^2 - \frac{dr^2}{1 - 2m/r} - r^2 (d\vartheta^2 + \sin^2 \vartheta d\phi^2).$$

We evaluate now the norm α of the four-acceleration along a congruence of absolute rest for Schwarzschild's manifold, that is accounted for in Hilbert's coordinates by (4.4) with $r > 2m$. It reads

$$(4.5) \quad \alpha = \left[\frac{m^2}{r^3(r - 2m)} \right]^{1/2}.$$

This local, invariant, intrinsic quantity diverges for $r \rightarrow 2m$. It defines a singularity that one meets when considering congruences of absolute rest closer and closer to the inner border of Schwarzschild manifold, i.e. closer and closer to the borders drawn in the interior of both the Hilbert and the Kruskal-Szekeres manifolds.

In this case the answer to the previous question is therefore simply: the border between the submanifolds endowed with different invariance groups of the just examined solutions is to be considered singular from a geometric standpoint, as soon as one does not limit the attention to the polynomial

invariants built with the Riemann tensor. As noticed long ago¹⁰ by Whittaker [45] and by Rindler [46], besides the geometrical meaning, α , and its singularity, have an immediate physical meaning too. Let us consider a test body of unit mass kept on a congruence of absolute rest by a dynamometer of negligible mass; also the other end of the dynamometer is assumed to follow a congruence of absolute rest. According to Whittaker and Rindler, the quantity α then equals the strength of the gravitational pull measured by the dynamometer [47].

Also the left and right quadrants of the so called boost-rotation symmetric solutions of footnote [7] are endowed, at each event, with a unique timelike Killing vector that is hypersurface-orthogonal with respect to a hypersurface of spacelike character¹¹. Therefore, this Killing vector too uniquely defines a direction of absolute rest, and from it a unique congruence of absolute rest is again obtained. Since the worldlines of the material particles of these solutions never cross the congruences of absolute rest, they can only be interpreted as worldlines of particles in a condition of absolute rest. Their current interpretation as worldlines of particles executing a uniformly accelerated motion with respect to an asymptotic reference system at spatial infinity is problematic [33], because it relies on an approximate asymptotic symmetry that contradicts the exact invariance group of the metric prevailing everywhere in the submanifolds of the left and right quadrants.

Like it happens in the Kruskal-Szekeres manifold, on crossing the boundaries between the left and right submanifolds and the upper and lower submanifolds of these solutions the unique timelike, hypersurface-orthogonal Killing vector becomes null and then spacelike. Let us calculate the norm α of the four-acceleration along congruences of absolute rest lying closer and closer to the boundaries of the left and right submanifolds with the upper and lower submanifolds of the solutions of footnote [7]. We can expect¹² that we shall find a local, invariant, intrinsic singularity of nonpolynomial kind, associated with the change of the invariance group that prevails there. This is indeed the case, as it was already shown in [33], to which the interested reader is referred for details.

In the Kerr-Newman solution, defined in Boyer-Lindquist coordinates by the interval (3.1), no unique, hypersurface-orthogonal, timelike Killing vector exists. However, a singular behaviour of α on approaching the boundary located at $r = r_0 = M + \sqrt{M^2 - J^2 - Q^2}$ can be invariantly proved as

¹⁰before Synge [48] eventually convinced the relativists that the wise plan is to forget about Newton's arrow and say "gravitational field = curvature of space-time".

¹¹In fact, the manifolds of these quadrants are diffeomorphic to the Weyl-Levi Civita manifolds [49, 50], for which the Killing group structure was examined in [44].

¹²in complete agreement with what occurs to the norm α of the four-acceleration calculated along a congruence of absolute rest in the left and right quadrants of the Kruskal manifold.

follows. Let λ and μ be two constants. The elements

$$(4.6) \quad \xi^k \frac{\partial}{\partial x^k} = \lambda \frac{\partial}{\partial t} + \mu \frac{\partial}{\partial \varphi}$$

of the Killing group prevailing for $r > r_0$ define invariantly a set of orbits. The squared norm of the first curvature on these orbits

$$(4.7) \quad \alpha^2 = -g_{ij} a^i a^j = -g_{ij} \left(\frac{\xi^i}{N} \right)_{;k} \frac{\xi^k}{N} \left(\frac{\xi^j}{N} \right)_{;l} \frac{\xi^l}{N}, \quad \text{with } N = \sqrt{g_{mn} \xi^m \xi^n},$$

contains always the factor $1/\Delta$ and diverges for orbits taken closer and closer to the surface $\Delta = 0$, for which $r = r_0 = M + \sqrt{M^2 - J^2 - Q^2}$. All Killing congruences defined by (4.6) are spacelike in the limit $\Delta \rightarrow 0$ except for the case given by $\mu = \lambda J / (r_0^2 + J^2)$. This congruence is timelike for $r > r_0 = M + \sqrt{M^2 - J^2 - Q^2}$ and null in the limit $r \rightarrow r_0$. The norm of its first curvature, i.e. the norm of its acceleration diverges in the same limit¹³. Hence, in the Kerr-Newman case, at the surface $r = r_0$ a local, invariant, intrinsic singularity is defined, despite the fact that the polynomial invariants built with the Riemann tensor are regular there.

5. CONCLUSION

We all learned that the Riemann curvature is the root of all scalar invariants that can be constructed at a certain event when only the metric in its infinitesimal neighbourhood is known. In the generic case (the case with trivial Killing group) this is not questioned. However, when the Killing group of a manifold is not trivial, its properties may produce local, intrinsic, invariant quantities without counterpart in quantities built with the polynomial invariants of the Riemann tensor.

Pasting together submanifolds endowed with nontrivial, physically different Killing groups, besides being a move not to be recommended per se, may produce a divergent behaviour when such invariant, intrinsic quantities are calculated at events closer and closer to the borders between the above mentioned submanifolds, even if the polynomial invariants of the Riemann tensor are not divergent there.

APPENDIX A. THE INFINITESIMAL KILLING VECTORS

A very simple definition of the infinitesimal Killing vectors is given by [7] and is reproduced here for the reader's convenience. Let us consider a pseudo Riemannian manifold equipped with two coordinate systems x'^i and x^i such that

$$(A.1) \quad x'^i = x^i + \xi^i,$$

¹³In the case of a static metric, $J = 0$, the congruence turns out to be the hypersurface-orthogonal one.

where ξ^i is an infinitesimal four-vector. Under this infinitesimal coordinate transformation, the components of the metric tensor g^{ik} in terms of g^{ik} read

$$(A.2) \quad g^{ik}(x^p) = \frac{\partial x'^i}{\partial x^l} \frac{\partial x'^k}{\partial x^m} g^{lm}(x^p) \approx g^{ik}(x^p) + g^{im} \frac{\partial \xi^k}{\partial x^m} + g^{km} \frac{\partial \xi^i}{\partial x^m}.$$

The quantities in the first and in the last term of (A.2) are calculated at the same event (apart from higher order infinitesimals). We desire instead to compare quantities calculated for the same coordinate value, i.e. evaluated at neighbouring events separated by the infinitesimal vector ξ^i . To this end, let us expand $g^{ik}(x^p + \xi^p)$ in Taylor's series in powers of ξ^p . By neglecting higher order infinitesimal terms, we can also substitute g^{ik} for g^{ik} in the term containing ξ^i of the expansion truncated at the first order term, and find:

$$(A.3) \quad g^{ik}(x^p) = g^{ik}(x^p) + g^{im} \frac{\partial \xi^k}{\partial x^m} + g^{km} \frac{\partial \xi^i}{\partial x^m} - \frac{\partial g^{ik}}{\partial x^m} \xi^m.$$

But the difference $\delta g^{ik}(x^p) = g^{ik}(x^p) - g^{ik}(x^p)$ has tensorial character and can be rewritten as

$$(A.4) \quad \delta g^{ik}(x^p) = \xi^{i;k} + \xi^{k;i}$$

in terms of the contravariant derivatives of ξ^i . When

$$(A.5) \quad \xi^{i;k} + \xi^{k;i} = 0$$

the metric tensor g^{ik} goes into itself under Lie's "Mitschleppen" [10]. An infinitesimal Killing vector is a four-vector ξ^i that fulfills (A.5).

REFERENCES

- [1] Einstein, A., (1915). *Sitzungsber. Preuss. Akad. Wiss., Phys. Math. Kl.*, 844 (submitted 25 Nov. 1915).
- [2] Hilbert, D., (1915). *Nachr. Ges. Wiss. Göttingen, Math. Phys. Kl.*, 395 (submitted 20 Nov. 1915).
- [3] Einstein, A., (1916). *Annalen der Physik* **49**, 769.
- [4] Kretschmann, E., (1917). *Annalen der Physik* **53**, 575.
- [5] Einstein, A., (1918). *Annalen der Physik* **55**, 241.
- [6] Frank, Ph., (1917). *Jahrbuch Forts. Math.* **46**, 1292.
- [7] Landau, L. and Lifshits, E., (1970). *Théorie des champs*, Éditions Mir, Moscou.
- [8] Klein, F. (1910). *Jhrber. d. d. Math. Vereinig.* **19**, 287.
- [9] Kretschmann, E., (1915). *Annalen der Physik* **48**, 907.
- [10] Schouten, J.A., (1954). *Ricci-calculus; an introduction to tensor analysis and its geometrical applications*, Springer, Berlin.
- [11] Klein, F., (1917). *Nachr. Ges. Wiss. Göttingen, Math. Phys. Kl.*, 469.
- [12] Noether, E., (1918). *Nachr. Ges. Wiss. Göttingen, Math. Phys. Kl.*, 235.
- [13] Schwarzschild, K., (1916). *Sitzungsber. Preuss. Akad. Wiss., Phys. Math. Kl.*, 189.
- [14] Hilbert, D., (1917). *Nachr. Ges. Wiss. Göttingen, Math. Phys. Kl.*, 53.
- [15] Antoci, S., and Liebscher, D.-E., (2003). *Gen. Relativ. Gravit.* **35**, 945.
- [16] Antoci, S., and Liebscher, D.-E., (2006). *General Relativity Research Trends*, Albert Reimer ed., pp. 177 - 213, Nova Science Publishers, New York. See also: <http://arxiv.org/abs/gr-qc/0406090>.
- [17] Kruskal, M.D., (1960). *Phys. Rev.* **119**, 1743.

- [18] Szekeres, G., (1960). *Publ. Math. Debrecen* **7**, 285.
- [19] Kerr, R. P., (1963). *Phys. Rev. Lett.* **11**, 237.
- [20] Newman, E. T., Couch, E., Chinnapared, K., Exton, A., Prakash, A., and Torrence, R., (1965). *J. Math. Phys.* **6**, 918.
- [21] Reissner, H., (1916). *Annalen der Physik* **50**, 106.
- [22] Nordström, G., (1918). *Proc. R. Acad. Amsterdam* **20**, 1238.
- [23] Bondi, H. (1957). *Rev. Mod. Phys.* **29**, 423.
- [24] Bonnor, W.B., and Swaminarayan, N.S., (1964). *Zeits. f. Phys.* **177**, 240.
- [25] Israel, W., and Khan, K.A., (1964). *Nuovo. Cim.* **33**, 331.
- [26] Bonnor, W.B., (1966). *Wiss. Zeits. Jena (Math-Nat. Reihe)* **15**, 71.
- [27] Bičák, J. (1968). *Proc Roy. Soc. A* **302**, 201.
- [28] Bičák, J., Hoenselaers, C., and Schmidt, B.G., (1983). *Proc. Roy. Soc. Lond. A* **390**, 397, 411.
- [29] Bičák, J., and Schmidt, B.G., (1984). *J. Math. Phys.* **25**, 600.
- [30] Bonnor, W.B., (1983). *Gen. Rel. Grav.* **15**, 535.
- [31] Bonnor, W.B., (1988). *Gen. Rel. Grav.* **20**, 607.
- [32] Bičák, J., and Schmidt, B.G., (1989). *Phys. Rev. D* **40**, 1827.
- [33] Antoci, S., Liebscher, D.-E., and Mihich, L., (2006). *Gen. Rel. Grav.* **38**, 15. See also: <http://arxiv.org/abs/gr-qc/0412102>.
- [34] Geroch, R., (1968). *J. Math. Phys.* **9**, 450.
- [35] Geroch, R., (1968). *Annals of Physics* **48**, 526.
- [36] Schmidt, B.G., (1971). *Gen. Rel. Grav.* **1**, 269.
- [37] Geroch, R., Kronheimer, E.H., and Penrose, R., (1972). *Proc. R. Soc. Lond. A* **327**, 545.
- [38] Ellis, G.F.R., and Schmidt, B.G. (1977). *Gen. Rel. Grav.* **8**, 915.
- [39] Thorpe, J.A., (1977). *J. Math. Phys.* **18**, 960.
- [40] Geroch, R., Liang Can-bin, and Wald, R.M., (1982). *J. Math. Phys.* **23**, 432.
- [41] Scott, Susan M., and Szekeres, P., (1994). *J. Geom. Phys.* **13**, 223.
- [42] Synge, J.L., (1950). *Proc. R. Irish Acad.* **53A**, 83.
- [43] Rindler, W., (2001). *Relativity, special, general and cosmological*, Oxford University Press, Oxford, pp. 265-267.
- [44] Ehlers, J., and Kundt, W., (1964). *Gravitation, An Introduction to Current Research*, L. Witten ed., Wiley, New York, pp. 49-101.
- [45] Whittaker, E. T., (1935). *Proc. R. Soc. London A* **149**, 384.
- [46] Rindler, W., (1960). *Phys. Rev.* **119**, 2082.
- [47] Antoci, S., Liebscher, D.-E., and Mihich, L., (2001). *Class. Quantum Grav.* **18**, 3463. Also: <http://arxiv.org/abs/gr-qc/0104035>.
- [48] Synge, J. L., (1966). *What is Einstein's Theory of Gravitation?*, in: Hoffman, B. (ed.), *Essays in Honor of Vaclav Hlavatý*, Indiana University Press, Bloomington, p. 7.
- [49] Weyl, H., (1917). *Annalen der Physik* **54**, 117.
- [50] Levi-Civita, T., (1919). *Rend. Acc. dei Lincei* **28**, 3.

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