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The Electromagnetic Properties of Material Media and Einstein's Unified Field Theory

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The issue of the form that the microscopic constitutive relation between electromagnetic inductions and fields should take in a dynamic spacetime is examined, and an answer that can be provided through the geometric objects of Einstein's unified field theory is considered. It is shown that the microscopic constitutive relation implied by these geometric objects allows to produce dynamically, by availing of fluctuations of the metric field with an appropriate behaviour, both the macroscopic relation between weak inductions and fields prevailing in vacuo and the one that occurs in nondispersive, nonconducting material media. The possible relevance of these results for a theory of matter in a dynamic spacetime that does not rely on the quantum framework is intimated, and conceptual reasons why it may be worth exploring such an alternative are outlined.

§ 1. Introduction

The essence of macroscopic electromagnetism is embodied by Maxwell's equations. Although they were originally conceived as pertaining to the flat space and to the absolute time of Newtonian heritage, the subsequent developments in the geometrical conception of spacetime have revealed how deep a position these equations occupy in the structure of physics. The notion of affine connection and the notion of metric are unnecessary for their existence; they can be written as soon as the primitive concept of unconnected four-dimensional manifold is introduced.[1] For, then, a contravariant skew tensor density $\mathbf{a}^{ik}$ and a covariant skew tensor b_{ik} can be defined, and we can write the tensor equations

$$\mathbf{a}^{is}{}_{,s}=4\pi\mathbf{j}^{i}\,,\quad b_{[ik,l]}=0\,,\tag{1}$$

where $\mathbf{j}^i$ is the conserved electric four-current density, the comma represents ordinary differentiation, and where we have assumed $b_{[ik,l]}\equiv(1/3)(b_{ik,l}+b_{kl,i}+b_{li,k})$. Of course, the spacetime description of physics requires that we supplement the primitive unconnected manifold with both an affine connection Γ^i_{kl} and with a metric s_{ik}; it is then natural to wonder how these new elements should enter the needed relation between $\mathbf{a}^{ik}$ and b_{ik} in order to best help the momentous task of accounting for the extremely varied phenomena displayed by electromagnetism in the material media. The current answer to this fundamental question has its remote origin in the electron theory developed by Lorentz when a dynamic conception of spacetime, involving a link between matter and geometry, still was a hope for the future. It supposes that the rich variety of behaviour that the relation between $\mathbf{a}^{ik}$ and b_{ik} displays in matter is a macroscopic deception. The truly important relation is the microscopic one; as such, it is unaccessible to our experience, but Nature has been so benign to provide us with a faithful replica in that special kind of macroscopic medium that we call vacuum. In that case, a quite simple relation between $\mathbf{a}^{ik}$ and b_{ik} can be assumed to hold; it involves the metric in algebraic form and reads

$$b_{ik}=a_{ik}\equiv(-s)^{-1/2}s_{ip}s_{kq}\mathbf{a}^{pq}\,, \tag{2}$$

where $s\equiv\det(s_{ik})$. Why not attempt to transfer this simple relation as it stands in the microscopic domain? While doing so, a further, substantial simplification can be introduced: We can totally disregard the dynamic character of spacetime and choose the coordinates so that $s_{ik}=\eta_{ik}\equiv\mathrm{diag}\,(-1,-1,-1,1)$. Once the laws that rule electromagnetism on a small scale are supposed known, and the dynamical laws obeyed by the elementary charges and currents that feel and produce the microscopic field are assigned, we can try to retrieve the whole richness of macroscopic electromagnetism in matter via statistical methods.

It is universally known that this program has met with an outstanding success, but only after it was recognized that Nature is not so unimaginative that we can simultaneously model the behaviour of electromagnetism on a small scale after macroscopic electromagnetism in vacuo, and borrow for the microscopic dynamics the very laws and concepts that hold on a large scale. In order to meet with the observed facts about macroscopic electromagnetism in matter, a new, intrinsically probabilistic mechanics is in fact adopted, and the goal of a thorough spacetime description of reality is thereby renounced, if we insist in attributing to microscopic electromagnetism the simple structure given by (1) and (2), with $s_{ik}=\eta_{ik}$. But also the direct physical meaning of this simple structure has to be renounced at last, for, in order to agree with experience, again with outstanding success, the method of field quantization has to be applied to the field equations (1) with the constitutive relation (2).

In writing these equations, curvilinear coordinates were here used, as it is required for evidencing the positions that they occupy in the structure of spacetime physics. Yet, the method of quantization can be confidently applied only when spacetime is assumed to be flat and Cartesian coordinates are adopted; as a long story of attempts intimates, quantum methods and concepts do not seem exactly in their own when confronted with the dynamic structure of spacetime. Therefore, despite the evident success of the current approach to electromagnetism in matter, it does not seem a complete waste of time if we reconsider the fundamental choice of (2) as constitutive relation for microscopic electromagnetism, which has determined all the subsequent developments. Equation (2) is extremely simple, but this simplicity means also that all the richness of structure that a dynamic spacetime can be conceived to exhibit on a microscopic scale is irrelevant to it. Due to this irrelevance, the whole burden of accounting for the manifold aspects of electromagnetism in matter is commited to the dynamics of elementary charges and currents. Now, we know from general relativity that the dynamical structure of spacetime actually rules essential properties of matter since, via the Ricci tensor, a dependence of the stress-momentum-energy tensor on the metric is established. Why should that structure be so idle with respect to another essential feature of matter, like the relation between electromagnetic inductions and fields?

§ 2. A constitutive relation for electromagnetism from Einstein's unified field theory

An alternative to the constitutive relation (2) is suggested by the later works of Einstein and of Schrödinger, devoted[2),3)] to the search of an extension of the general relativity of 1915 that could provide a dynamical description of spacetime encompassing both gravitation and electromagnetism. We aim at expressing the properties of the material continuum by means of the geometric objects of the non-Riemannian structure considered by Einstein. The four-dimensional manifold with real coordinates x^i is therefore endowed with a nonsymmetric tensor density $\boldsymbol{g}^{ik}$ and with a nonsymmetric affine connection Γ^i_{kl}, and the following definitions of the material properties are introduced:[4)]

$$\boldsymbol{g}^{qr}{}_{,p}+\boldsymbol{g}^{sr}\Gamma^q_{sp}+\boldsymbol{g}^{qs}\Gamma^r_{ps}-\boldsymbol{g}^{qr}\Gamma^t_{(pt)}=(4\pi/3)(\boldsymbol{j}^q\delta_p{}^r-\boldsymbol{j}^r\delta_p{}^q)\,, \tag{3}$$

$$\boldsymbol{g}^{[is]}{}_{,s}=4\pi\boldsymbol{j}^i\,, \tag{4}$$

$$B_{(ik)}(\Gamma)=8\pi(T_{ik}-(1/2)s_{ik}s^{pq}T_{pq})\,, \tag{5}$$

$$B_{[[ik],l]}(\Gamma)=(8\pi/3)K_{ikl}\,. \tag{6}$$

The affine connection Γ^i_{kl}, by definition constrained to yield $\Gamma^k_{[ik]}=0$, is used to build the symmetrized Ricci tensor[5)]

$$B_{ik}(\Gamma)=\Gamma^a_{ik,a}-(1/2)(\Gamma^a_{ia,k}+\Gamma^a_{ka,i})-\Gamma^a_{ib}\Gamma^b_{ak}+\Gamma^a_{ik}\Gamma^b_{ab}\,, \tag{7}$$

while the role of metric is attributed[6)] to the symmetric tensor s_{ik}, defined by

$$\boldsymbol{s}^{ik}=\boldsymbol{g}^{(ik)}\,,\quad \boldsymbol{s}^{ik}=(-s)^{1/2}s^{ik}\,,\quad s^{il}s_{kl}=\delta_k{}^i\,,\quad s=\det(s_{ik})\,. \tag{8}$$

The contracted Bianchi identities[6),4)] read

$$\mathcal{T}_k{}^l{}_{;l}=(1/2)(\boldsymbol{j}^iB_{[ki]}+K_{ikl}\boldsymbol{g}^{[li]})\,, \tag{9}$$

where the semicolon denotes covariant differentiation with respect to the Christoffel affine connection

$$\Sigma^i_{kl}=(1/2)s^{im}(s_{mk,l}+s_{ml,k}-s_{kl,m}); \tag{10}$$

here and in the following indices are raised and lowered with s^{ik} and s_{ik}, tensor densities are built with $(-s)^{1/2}$, etc. We interpret T_{ik} as the energy tensor, $\boldsymbol{j}^i$ as the electric current density, K_{ikl} as the magnetic current; Eqs. (3)~(6) then define the material properties of an electromagnetic medium, in which $\boldsymbol{g}^{[ik]}$ represents the electric induction and the magnetic field, while $B_{[ik]}(\Gamma)$ represents the electric field and the magnetic induction. Through these definitions the constitutive relation between inductions and fields is made akin to the relation between the metric and the energy tensor, and is inextricably entwined with it: For a given $\boldsymbol{g}^{ik}$, both are produced in one go by solving (3) for Γ^i_{kl} and by substituting the latter in $B_{ik}(\Gamma)$. The essential role played by the Riemannian structure entailed by the metric s_{ik} on the non-Riemannian continuum that we are considering is evident from (9), for only when that structure is introduced the form of the contracted Bianchi identities becomes clear and physically transparent. Equation (9) asserts that the force density of electromagnetic nature due to the Lorentz coupling of $\boldsymbol{j}^i$ to $B_{[ik]}(\Gamma)$ and of K_{ikl} to $\boldsymbol{g}^{[ik]}$, defined in the non-Riemannian manifold under consideration, is found responsible for the nonconservation of the energy tensor T_{ik} of matter defined by (5) as soon as the manifold is further equipped with the metric s_{ik} according to the choice expressed by (8), and with the Riemannian spacetime structure engendered by it.

§ 3. The constitutive relation for weak inductions and fields and the small scale behaviour of the metric

We have drawn from Einstein's later work a set of definitions for the properties of the material continuum in terms of geometrical objects that implies inter alia a completely new form of the constitutive relation of the electromagnetic medium, since the expression of $B_{[ik]}$, that represents the electric field and the magnetic induction, when written in terms of $\boldsymbol{g}^{ik}$, is homogeneous of degree two with respect to differentiation,[7] i.e., it consists of an aggregate of terms, which are either linear in the second derivatives of $\boldsymbol{g}^{ik}$ or quadratic in the first derivatives of $\boldsymbol{g}^{ik}$, as it occurs for the expression of the energy tensor. Exhibiting the opportunities offered by this new form is an extremely demanding task: The preliminary move of solving (3) for the affine connection[8),9)] already leads to unsurveyable expressions. Let us set

$$\boldsymbol{g}^{[ik]}=\boldsymbol{a}^{ik}, \quad \boldsymbol{a}^{ik}=(-s)^{1/2}a^{ik}, \quad a_{ik}=s_{ip}s_{kq}a^{pq}, \tag{11}$$

and assume that, while $\boldsymbol{s}^{ik}$ is arbitrary, $\boldsymbol{a}^{ik}$ and its derivatives are so small that can be treated as first order infinitesimal quantities. The linear approximation to $B_{[ik]}$ then reads[4)]

$$B_{[ik]}=(2\pi/3)(j_{i,k}-j_{k,i})+(1/2)(a_i{}^nS_{nk}-a_k{}^nS_{ni}+a^{pq}S_{pqik}+a_{ik}{}_{;a}{}^{;a}), \tag{12}$$

where

$$S^i{}_{klm}=\Sigma^i_{kl,m}-\Sigma^i_{km,l}-\Sigma^i_{al}\Sigma^a_{km}+\Sigma^i_{am}\Sigma^a_{kl} \tag{13}$$

is the Riemann tensor of s_{ik}, $S_{ik}\equiv S^p{}_{ikp}$ is its Ricci tensor and the "contravariant derivative" notation $a_{ik}{}^{;l}\equiv s^{lm}a_{ik;m}$ is used. Equation (12) says that the relation between inductions and fields depends on the short range behaviour of $\boldsymbol{s}^{ik}$ and of $\boldsymbol{a}^{ik}$ in an essential way. Let us concentrate at present on the small scale features of $\boldsymbol{s}^{ik}$. What do we know about the short range behaviour of the metric? We know, of course, that if we probe the metric with macroscopic devices that surely do not convey information about a single event x^i, but provide us with a sort of average information about a sizable spacetime region Ω, a suitable coordinate system exists, with respect to which the probed entity, let us call it $\bar{s}_{ik}$, is extremely close to the Minkowski metric η_{ik}, and its derivatives are extremely small. One may well hope that the very concept of a metric field s_{ik}, a mathematical abstraction from the macroscopic experience, may survive the transition from the large to the small scale, but we should not be so exacting as to require that all the properties of the macroscopically probed metric $\bar{s}_{ik}$ belong also to s_{ik}.

We can for instance assume that everywhere s_{ik} departs very slightly from its macroscopic average $\bar{s}_{ik}$, and that it does so through fluctuations[10] whose characteristic length is very short when compared with the spacetime extension of the devices that probe $\bar{s}_{ik}$, so that its derivative $s_{ik,l}$ is by no means small. To proceed further, we need to formalize the notion of macroscopic spacetime average in a covariant theory.[11] Let us assume that $O^{lm::}_{pq..}(x^i)$ is a geometric object whose components exhibit a fluctuating behaviour in the coordinate system x^i. We endow a generic event $x_0{}^i$ with a spacetime neighbourhood $\Omega(x_0{}^i)$ that includes a very large number of ripples of $O^{lm::}_{pq..}(x^i)$. We indicate the spacetime average of $O^{lm::}_{pq..}(x^i)$ associated to the neighbourhood $\Omega(x_0{}^i)$ as $\langle O^{lm::}_{pq..}(x^i)\rangle_{\Omega(x_0{}^i)}$ or simply as $\bar{O}^{lm::}_{pq..}(x^i)$, and we pose

$$\bar{O}^{lm::}_{pq..}(x_0{}^i)=\int_{\Omega(x_0{}^i)}O^{lm::}_{pq..}(x^i)d\Omega\Big/\int_{\Omega(x_0{}^i)}d\Omega\,. \tag{14}$$

In the coordinate system x^i, a prescription can be given for attributing a neighbourhood $\Omega(x^i)$ to each event within a given spacetime region by assuming that, if $\Omega(x_0{}^i)$ is the neighbourhood pertaining to $x_0{}^i$, the neighbourhood associated with $x_0{}^i+\delta x^i$ contains the points whose coordinates are obtained by giving to the coordinates of the points in $\Omega(x_0{}^i)$ the increment δx^i. The average field $\bar{O}^{lm::}_{pq..}(x^i)$ can thus be defined in the given spacetime region, and with respect to the coordinate system x^i; the definition is such that

$$\langle O^{lm::}_{pq..,r}\rangle_{\Omega(x^i)}=\bar{O}^{lm::}_{pq..,r}(x^i)\,. \tag{15}$$

Of course, the average field $\bar{O}^{lm::}_{pq..}(x^i)$ does not transform as a geometrical object with respect to an arbitrary coordinate transformation $x'^i=f^i(x^k)$, and the prescription for associating neighbourhoods to events is not retained in the primed coordinate system. However, a subset of coordinate transformations $x'^i=h^i(x^k)$ can be considered, such that, if the functions h^i and their derivatives are expanded in Taylor's series around a generic event $x_0{}^i$:

$$x'^i=h^i(x_0{}^k)+(h^i{}_{,m})_0(x^m-x_0{}^m)+(1/2)(h^i{}_{,m,n})_0(x^m-x_0{}^m)(x^n-x_0{}^n)+\cdots\,,$$

$$x'^i{}_{,m}=(h^i{}_{,m})_0+(h^i{}_{,m,n})_0(x^n-x_0{}^n)+\cdots\,, \tag{16}$$

etc., the leading term of each expansion is much larger than the subsequent ones for all the events x^i within the neighbourhood $\Omega(x_0{}^i)$. With respect to this subset of transformations $\bar{O}^{lm::}_{pq..}(x^i)$ behaves, with the approximation deriving from the above hypotheses, as a geometric object endowed with the same transformation law as the one possessed by $O^{lm::}_{pq..}(x^i)$ with respect to the general transformations. This subset is sufficient for reckoning with the changes of reference frame that can be set up in the world of macroscopic experience, to which the averages refer. We call henceforth macroscopic a transformation of coordinates belonging to the subset defined above, and macroscopic will also be called a coordinate system x'^i reachable from x^i through a macroscopic coordinate transformation. We find that the prescription for defining adjacent neighbourhoods $\Omega(x^i)$ and $\Omega(x^i+\delta x^i)$, as well as (15), hold with the previously mentioned approximation in all the macroscopic coordinate systems.

§ 4. Fluctuations of the metric that produce the constitutive relation for the macroscopic vacuum

Having shown how spacetime averages can be defined in a generally covariant theory, a statistical approach to the constitutive relation for macroscopic electromagnetism becomes possible. Equation (12) holds when $\mathbf{a}^{ik}$ is small and slowly varying, while $\mathbf{s}^{ik}$ is arbitrary. As suggested by Lanczos, we assume that s_{ik} exhibits fluctuations of a very small amplitude with a very short characteristic length around its average value $\bar{s}_{ik}$, and look for the average field $\bar{B}_{[ik]}$ that, in the linear approximation of (12), corresponds to the weak and slowly varying $\mathbf{a}^{ik}$. Equation (12) can be expressed as the sum of several addenda, singly written as the product of either $\mathbf{a}^{ik}$, or $\mathbf{a}^{ik}{}_{,l}$, or else $\mathbf{a}^{ik}{}_{,l,m}$, times other factors, each one individually given by s^{ik}, s_{ik}, $(-s)^{-1/2}$ and by the ordinary derivatives of s_{ik} up to second order. In the addenda where s_{ik} is differentiated twice no other derivatives appear. We can write those addenda as the overall derivative of a product that contains only one first derivative of s_{ik}, minus the sum of terms in which the product of two first derivatives of s_{ik} may occur. Due to (15), the whole averaging of $B_{[ik]}$ reduces thus to the averaging of individual terms that contain only products, and in which the derivatives of s_{ik} do not exceed the first degree of differentiation. While averaging the individual term, we shall look at the number of factors $s_{ik,l}$ appearing in it. Due to the assumed smallness of the fluctuations, we can set $\langle s_{ik}s_{lm}\rangle=\bar{s}_{ik}\bar{s}_{lm}$ and $\langle s_{ik}s_{lm,n}\rangle=\bar{s}_{ik}\bar{s}_{lm,n}$. Therefore the average of a term in which $s_{ik,l}$ does not appear, or appears once, is equal to the product of the averages of the single factors. The remaining terms contain the product $s_{ik,l}s_{mn,p}$, whose average of course is not equal to $\bar{s}_{ik,l}\bar{s}_{mn,p}$. Since the fluctuations have very small amplitude, the quantity $F_{iklmnp}\equiv\langle s_{ik,l}s_{mn,p}\rangle-\bar{s}_{ik,l}\bar{s}_{mn,p}$ behaves as a tensor with respect to the macroscopic coordinate transformations; it encodes the statistical information about the fluctuating s_{ik} needed to complete the evaluation of $\bar{B}_{[ik]}$.

Let us consider the form taken by F_{iklmnp} when the fluctuations are such that s_{ik} is conformally related to its average:

$$s_{ik}=e^{\sigma}\bar{s}_{ik}\,, \qquad |\sigma|\ll 1\,; \tag{17}$$

we get

$$F_{iklmnp}=\bar{s}_{ik}\bar{s}_{mn}\langle e^{2\sigma}\sigma_{,l}\sigma_{,p}\rangle=\bar{s}_{ik}\bar{s}_{mn}c_{lp}\,, \tag{18}$$

where $c_{lp}=c_{pl}$ behaves as a tensor under the macroscopic transformations. We note that (12) can be rewritten as

$$B_{[ik]}=(2\pi/3)(j_{i,k}-j_{k,i})+(1/2)(a^{pq}C_{pqik}+(1/3)Sa_{ik}+a_{ik}\,{}_{;a}^{\;\,a})\,, \tag{19}$$

where $S\equiv s^{pq}S_{pq}$ and

$$C_{ijkl}\equiv S_{ijkl}-(1/2)(S_{jk}s_{il}+s_{jk}S_{il}-S_{jl}s_{ik}-s_{jl}S_{ik})+(S/6)(s_{jk}s_{il}-s_{jl}s_{ik}) \tag{20}$$

is Weyl's conformal curvature tensor; due to (17), we have that $C_{ijkl}(s_{ab})=e^{\sigma}C_{ijkl}(\bar{s}_{ab})$. The calculation of $\bar{B}_{[ik]}$ is outlined in Appendix I; in it the notation "|" for the covariant differentiation with respect to $\Sigma^i_{kl}(\bar{s}_{ab})$ has been introduced. The result reads

$$\langle B_{[ik]}(s_{ab}, a_{ab})\rangle = B_{[ik]}(\bar{s}_{ab}, \bar{a}_{ab}) + D\bar{a}_{ik}\,, \tag{21}$$

where the function $B_{[ik]}(s_{ab}, a_{ab})$ is defined by (19), and $D = -(3/2)\bar{s}^{pq}c_{pq}$. We see that $\bar{B}_{[ik]}$ is constituted by two terms; the first one displays on the average fields $\bar{s}_{ik}$ and $\bar{a}_{ik}$ the same dependence that $B_{[ik]}$ has on the unaveraged ones s_{ik} and a_{ik}, while the second one is just given by $\bar{a}_{ik}$ times a factor D that behaves as a scalar under macroscopic coordinate transformations. The sign and the value of D are ruled by the conformal fluctuations of the metric in the way shown by (18). Let us assume that in a given region of spacetime D is constant, and that the fluctuations of s_{ik} have so short a characteristic length that the first term on the right-hand side of (21) is totally negligible with respect to the second one. In that region the macroscopic constitutive relation for electromagnetism, i.e., the relation between average inductions and average fields, will take the form of Eq. (2), that experience says to apply to the macroscopic vacuum, although the microscopic relation entailed by (12) has an entirely different character. If the average magnetic current $\bar{K}_{ikl}$ is supposed to vanish, as one requires in macroscopic electromagnetism, we find from (6) and (21), thanks to (15):

$$\langle B_{[[ik],l]}\rangle = \bar{B}_{[[ik],l]} = B_{[[ik],l]}(\bar{s}_{ab}, \bar{a}_{ab}) + (D\bar{a}_{[ik]})_{,l]} = 0\,; \tag{22}$$

within the considered region the fields $\boldsymbol{a}^{ik}$, $\bar{B}_{[ik]}$ will just behave as expected to occur to macroscopic inductions and fields in vacuo. Moreover, since both $\boldsymbol{j}^i$ and $\boldsymbol{g}^{[ik]}$ were assumed to be slowly varying, we have

$$\langle \boldsymbol{j}^i B_{[ki]} + K_{ikl}\boldsymbol{g}^{[li]}\rangle = \boldsymbol{j}^i\bar{B}_{[ki]} + \bar{K}_{ikl}\boldsymbol{g}^{[li]}\,, \tag{23}$$

and the average of the right-hand side of the conservation identity (9) acquires the form appropriate to the force density exerted on a macroscopic electric four-current in vacuo.

Let us consider a macroscopic coordinate system for which at a given event $\bar{s}_{ik} = \eta_{ik}$, as well as the coordinate systems that can be reached from this through a Lorentz transformation. It is remarkable that the dynamical structure that we have attributed to the metric s_{ik} provides us, when D is constant, with a Lorentz invariant vacuum as far as the propagation of macroscopic electromagnetic fields is concerned, despite the fact that in the coordinate systems considered above $F_{iklmnp} = \bar{s}_{ik}\bar{s}_{mn}c_{lp}$ does not behave, at the chosen event, as a Lorentz invariant quantity, unless $c_{lp} = K\bar{s}_{lp}$, where K is a constant. This noninvariance is apparent also in the macroscopic background energy density produced by the conformally fluctuating metric. When $\boldsymbol{a}^{ik} = 0$ we find in fact

$$\begin{aligned}8\pi\langle \boldsymbol{\mathfrak{I}}_k{}^l\rangle &= \langle \boldsymbol{s}^{il}[S_{ik}(s_{ab}) - (1/2)s_{ik}S(s_{ab})]\rangle \\ &= (-\bar{s})^{1/2}\bar{s}^{il}[S_{ik}(\bar{s}_{ab}) - (1/2)\bar{s}_{ik}S(\bar{s}_{ab})] \\ &\quad + (3/2)(-\bar{s})^{1/2}[\bar{s}^{il}c_{ik} - (1/2)\delta_k{}^l\bar{s}^{pq}c_{pq}]\,;\end{aligned} \tag{24}$$

the average density $\langle \mathfrak{J}_k{}^l \rangle$ is constituted by the sum of two terms: The first one represents the energy tensor density that is prescribed by the macroscopic metric $\bar{s}_{ik}$, while the second one contains the contribution coming from the fluctuations. Like F_{iklmnp}, also this second term, when considered in the coordinate systems for which locally $\bar{s}_{ik}=\eta_{ik}$, is not a Lorentz invariant quantity unless $c_{ik}=K\bar{s}_{ik}$. When the latter occurrence is verified, in a generic macroscopic coordinate system we have

$$8\pi\langle \mathfrak{J}_k{}^l(s_{ab})\rangle = 8\pi \mathfrak{J}_k{}^l(\bar{s}_{ab}) - (3/2)K(-\bar{s})^{1/2}\delta_k{}^l\,, \tag{25}$$

and the term due to the fluctuations takes the qualitative form appropriate to account for the vacuum energy and pressure content of the continuum intimated by quantum field theory considerations,[12] while the macroscopic metric $\bar{s}_{ik}$ can still display a nearly Minkowskian behaviour, in agreement with macroscopic experience. According to (18) the condition $c_{ik}=K\bar{s}_{ik}$, where K is real, requires for its fulfillment a metric field s_{ik} that exhibits complex fluctuations around its real average $\bar{s}_{ik}$. In Einstein's unified field theory $\boldsymbol{s}^{ik}$ was assumed real, while $\boldsymbol{a}^{ik}$ can be chosen to be either real or pure imaginary according to whether the real nonsymmetric or the complex Hermitian version of the theory is adopted.[2] From the mathematical standpoint it is however conceivable that both $\boldsymbol{s}^{ik}$ and $\boldsymbol{a}^{ik}$ assume complex values, and the macroscopic physical interpretation is not spoiled if the macroscopic averages turn out with the appropriate reality properties; this fact occurs in the case under consideration if the constant K is real valued, as assumed. We note that a result closely similar to the one displayed by (21) and (25) is achieved[11] if one considers nonconformal complex fluctuations of s_{ik} such that

$$F_{iklmnp} = K'\,\bar{s}_{lp}(\bar{s}_{im}\bar{s}_{kn} + \bar{s}_{in}\bar{s}_{km})\,, \tag{26}$$

where K' is a real constant; also in this case the continuum allows for the propagation of macroscopic electromagnetic fields as it occurs in a Lorentz invariant vacuum, and exhibits a background energy and pressure content like the one suggested for vacuum by quantum field theory. The case of the conformal fluctuations considered here seems interesting, since it shows that the Lorentz invariance of F_{iklmnp} and of the background energy tensor density when $\bar{s}_{ik}=\eta_{ik}$ locally is not a prerequisite for producing a dynamical vacuum in which macroscopic electromagnetic fields propagate in a Lorentz invariant manner. One should not forget that while the constancy of the velocity of electromagnetic disturbances in vacuo is a cogent experimental fact, the Lorentz invariance of the energy and pressure content of vacuum is a more formal assumption, whose necessity within a given theory can be proved only indirectly, through the agreement with observed facts of the physical behaviour that the said theory attributes to "real", nonvacuum matter.

§ 5. Fluctuations of the metric and the constitutive relation of material nonconducting media

The introduction of a metric field that fluctuates in the manners considered above allows to produce, by starting from the microscopic relation (12), a constitutive relation for macroscopic electromagnetism appropriate to the vacuum in the case of weak inductions and fields. The way kept in achieving this result naturally suggests exploring how the constitutive relation comes out if we introduce fluctuations of the metric which exhibit a lesser degree of symmetry. Can we avail of such fluctuations for reproducing the wide variety of behaviour that the macroscopic constitutive relation exhibits in material nonconducting media? Let us try to get a first answer through a simple example, and consider a region of spacetime for which, in a given coordinate system, the average metric $\bar{s}_{ik}$ is everywhere equal to the Minkowski metric η_{ik}, and the metric s_{ik} is such that $s_{44}=1$, $s_{\lambda 4}=s_{4\lambda}=0$, while its spatial part $s_{\lambda\mu}$ exhibits small conformal fluctuations around $\eta_{\lambda\mu}$ with short characteristic length:

$$s_{\lambda\mu}=e^{\sigma}\eta_{\lambda\mu}\,, \quad |\sigma|\ll 1\,. \tag{27}$$

Greek letters henceforth denote the spatial coordinates; the function σ is assumed to depend on the four coordinates x^i; from (27) it follows that the nonvanishing components of F_{iklmnp} read

$$F_{\lambda\mu a\rho\sigma b}=\eta_{\lambda\mu}\eta_{\rho\sigma}\langle e^{2\sigma}\sigma_{,a}\sigma_{,b}\rangle=\eta_{\lambda\mu}\eta_{\rho\sigma}c_{ab}\,. \tag{28}$$

Let us look for the average field $\bar{B}_{[ik]}$ that, again in the linear approximation of (12), corresponds to a weak and slowly varying $\boldsymbol{a}^{ik}$. The calculation is outlined in Appendix II; we find

$$\begin{aligned}\bar{B}_{[\lambda\mu]}&=(2\pi/3)(\bar{j}_{\lambda,\mu}-\bar{j}_{\mu,\lambda})+(1/2)\,\bar{a}_{\lambda\mu;a}{}^{;a}\\&\quad+(1/8)[\,\bar{a}_{\lambda}{}^{\varepsilon}c_{\mu\varepsilon}-\bar{a}_{\mu}{}^{\varepsilon}c_{\lambda\varepsilon}-\bar{a}_{\lambda}{}^{4}c_{\mu 4}+\bar{a}_{\mu}{}^{4}c_{\lambda 4}-5\,\bar{a}_{\lambda\mu}\eta^{pq}c_{pq}]\,,\\ \bar{B}_{[4\mu]}&=(2\pi/3)(\bar{j}_{4,\mu}-\bar{j}_{\mu,4})+(1/2)\,\bar{a}_{4\mu;a}{}^{;a}\\&\quad+(1/8)[\,\bar{a}_{\mu}{}^{\varepsilon}c_{4\varepsilon}-3\,\bar{a}_{4}{}^{\varepsilon}c_{\mu\varepsilon}-\bar{a}_{4\mu}(9\eta^{\gamma\delta}c_{\gamma\delta}+12\eta^{44}c_{44})]\,.\end{aligned} \tag{29}$$

Again, we can assume that the fluctuations of the metric have so short a characteristic length that the terms on the right-hand sides of (29) where c_{ik} does not appear are negligible with respect to the remaining ones. In that case (29) expresses a constitutive relation for electromagnetism as it is appropriate to a linear, nondissipative, spatially anisotropic, nonreciprocal, nonuniform, nondispersive and nonconducting medium.[13] Let us restrict the coefficients c_{ik} defined in (28) by assuming that, in the chosen coordinate system

$$c_{\lambda\mu}=\alpha\eta_{\lambda\mu}\,, \quad c_{\lambda 4}=0\,, \quad c_{44}=\beta\,. \tag{30}$$

Then (29) reduces to

$$\bar{B}_{[\lambda\mu]}=-(1/8)(13\alpha+5\beta)\,\bar{a}_{\lambda\mu}\,, \quad \bar{B}_{[4\mu]}=-(1/8)(30\alpha+12\beta)\,\bar{a}_{4\mu}\,, \tag{31}$$

if the terms on the right-hand sides of (29) where c_{ik} does not appear are omitted as negligible. If α and β do not depend on the coordinates x^i, (31) provides the constitutive relation for a linear, nondissipative, isotropic, reciprocal, uniform, nondispersive and nonconducting medium. Through the identification

$$\begin{aligned}&(\bar{B}_{[14]},\,\bar{B}_{[24]},\,\bar{B}_{[34]})=\boldsymbol{E}\,, \quad (\bar{B}_{[23]},\,\bar{B}_{[31]},\,\bar{B}_{[12]})=\boldsymbol{B}\,,\\&(\bar{a}_{14},\,\bar{a}_{24},\,\bar{a}_{34})=\boldsymbol{D}\,, \qquad (\bar{a}_{23},\,\bar{a}_{31},\,\bar{a}_{12})=\boldsymbol{H}\,,\end{aligned} \tag{32}$$

we can rewrite (31) as

$$B=-(1/8)(13\alpha+5\beta)H\,,\quad E=-(1/8)(30\alpha+12\beta)D\,. \tag{33}$$

The velocity v for the propagation of macroscopic electromagnetic disturbances in this medium is given by

$$v=\left(\frac{30\alpha+12\beta}{13\alpha+5\beta}\right)^{1/2}. \tag{34}$$

Let us assume $\alpha<0$, $\beta>0$, a choice that corresponds to real fluctuations of the metric. Then the velocity v turns out to be real and less than unity if

$$2\beta/5<-\alpha<7\beta/17\,. \tag{35}$$

What is the average tensor density $\langle \mathfrak{T}_k{}^l\rangle$ associated with the media for which (29) or (31) hold? Since $\boldsymbol{a}^{ik}$ is small and slowly varying, let us calculate it for the case $\boldsymbol{a}^{ik}=0$. For the anisotropic, nonreciprocal medium the components of $8\pi\langle \mathfrak{T}_k{}^l\rangle=\langle(-s)^{1/2}[s^{il}S_{ik}-(1/2)\delta_k{}^l S]\rangle$ are

$$\begin{aligned}
&8\pi\langle \mathfrak{T}_\mu{}^\nu\rangle=(1/2)\eta^{\lambda\nu}c_{\lambda\mu}-(1/4)\delta_\mu{}^\nu(\eta^{\alpha\beta}c_{\alpha\beta}+3\eta^{44}c_{44})\,,\\
&8\pi\langle \mathfrak{T}_\mu{}^4\rangle=(3/2)\eta^{44}c_{4\mu}\,,\quad 8\pi\langle \mathfrak{T}_4{}^\mu\rangle=(1/2)\eta^{\mu\lambda}c_{4\lambda}\,,\\
&8\pi\langle \mathfrak{T}_4{}^4\rangle=-(1/4)\eta^{\alpha\beta}c_{\alpha\beta}+(3/4)\eta^{44}c_{44}\,.
\end{aligned} \tag{36}$$

If the medium is made isotropic and reciprocal through the additional positions (30), the nonvanishing components of $8\pi\langle \mathfrak{T}_k{}^l\rangle$ reduce to

$$8\pi\langle \mathfrak{T}_\mu{}^\nu\rangle=-(1/4)\delta_\mu{}^\nu(\alpha+3\beta)\,,\quad 8\pi\langle \mathfrak{T}_4{}^4\rangle=(3/4)(\beta-\alpha)\,. \tag{37}$$

When $\alpha<0$, $\beta>0$ and the inequalities (35), ensuring that the velocity v of the electromagnetic disturbances is real and less than unity, are satisfied, the nonvanishing components of $\langle \mathfrak{T}_k{}^l\rangle$ happen to occur with the form appropriate to a medium endowed with positive energy density and with positive isotropic pressure,[14] if the dimensional constant needed for translating $\mathfrak{T}_k{}^l$ from the geometrical units to the physical ones is assumed positive. As regards the sign and the value of this dimensional constant, we emphasize that there is no compelling reason, in the present context, for adhering to the choice that is done in general relativity, a choice essentially dictated by the desire to impress a neo-Newtonian interpretation on that theory.

§ 6. Perspectives

The results of the previous section raise delicate questions of concept and of method. Can we really conceive building a macroscopic body out of a fluctuating $\boldsymbol{s}^{ik}$ or, more generally, out of a fluctuating $\boldsymbol{g}^{ik}$? This proposal closely adheres to the original program of general relativity, that aims at an objective description of reality in space and time in terms of geometric entities, and that today is usually dismissed as unrealistic. The main reason for this pessimistic attitude is to be searched in the position that quantum theory has come to occupy in the conceptual structure of physics: Originally born as a set of rules for a new microscopic description of charged particles that could account for the observed facts while preserving the assumptions for microscopic electromagnetism done by Lorentz, it has developed into an overall framework that is confidently used for the description of all microscopic processes, whatever their nature, and for predicting the macroscopic properties of matter in terms of its microscopic behaviour. This framework consists in a system of concepts and rules of a general, abstract character; as such, it does not constitute a physical theory per se, but it can generate a physical theory for a given class of phenomena through a two-steps procedure. First a theory, possibly a field theory, is formulated for giving abstract mathematical representation to the microscopic entities that one aims at describing and to their interactions; although this theory is either borrowed from classical physics or modeled after its pattern, it is not intended as possessing a direct physical meaning, but as a preliminary input structure on which the quantum theoretical rules operate to provide the final, physically relevant theory. It has thus become customary to assess the value of a given field theory according to its usefulness as input structure for the quantization process.

When considered from this standpoint, general relativity appears in an odd position, for it looks both useless and a challenging issue at the same time. It seems useless because, according to estimates that view it as the field theory of gravitation that has replaced Newton's theory and contains the latter in a certain limit, it looks quantitatively irrelevant to the description of atomic phenomena, at least in ordinary circumstances: Why worry about the quantization of gravitation, if the Newtonian attraction between two charged elementary particles, according to the naive extrapolation to a microscopic scale of macroscopic concepts and laws, is so exceedingly small when compared to their Coulomb interaction? Yet, despite this hint of quantitative irrelevance, if we do not find the way for applying the axiomatic structure of quantum field theory to general relativity, the very role of quantum concepts and methods as the overall framework within which physical theories have to be formulated remains in doubt. Therefore, and although no experimental evidence imposes confronting this challenge, strong reasons of principle have given origin to a long and up to now unsuccessful struggle for bridging the gap between quantum theory and general relativity. A significant evolution of mood has accompanied this struggle; it can be described in retrospect as a progressive retreat from a bold faith in the applicability of the concepts and of the axiomatic structure of the former theory to the latter towards a more thoughtful attitude, imposed by the ever growing consciousness of the formidable difficulties and of the intricate conceptual problems that such a presumption of applicability brings with itself.[15] With the lapse of time a notion deeply felt in the early days of general relativity and subsequently obscured to a certain degree is reinstating itself with great strength: General relativity is not just a field theory for the gravitational interaction, as such on an equal footing with other theories describing other fields; it is primarily a dynamical theory of spacetime itself. The postulates of quantum theory have been originally chosen for providing a description of microscopic phenomena in the absolute space and with respect to the absolute time of Newton, and have been subsequently adapted to deal with the rigid Minkowski background of special relativity; if spacetime has a dynamic character, as general relativity forcefully intimates, we cannot limit ourselves to reconsidering the validity of these postulates for the description of gravity on a microscopic scale: They need to be reexamined also in the context of atomic and subatomic physics, where they have met with such an unconditional success as to efface, in the minds of most physicists, many conceptual problems posed by their introduction. We shall inquire whether we have been forced to the adoption of such postulates because we have failed to appreciate the influences that a dynamic structure of spacetime can have on the microscopic behaviour of matter, and we shall try to develop a theory of matter in which the dynamic character of spacetime is kept into account since the very beginning. The resolution to develop such a theory without the help of the quantum theoretical framework may seem odd and unreasonable, in view of many successes reported by the quantum methods in the description of matter, but it may well happen that this will prove to be the only viable alternative in the long run, if the attempts to encompass general relativity within the quantum framework do not eventually bring to concrete results. The geometrical objects of Einstein's unified field theory offer a possible startpoint for pursuing this alternative; of course, one cannot know in advance whether they are really adequate to such a task, although the versatility displayed by the new constitutive relation for microscopic electromagnetism, as well as the close connection posed between this relation and the one prevailing between the metric and the energy tensor seem promising features of immediate physical interest.

In the present work some statistical properties of the small scale behaviour of the metric field s_{ik} have been prescribed a priori, and some macroscopic consequences of these assumptions have been looked for, in the case when $\boldsymbol{a}^{ik}$ is weak and slowly varying. Through this approach it is not possible to go beyond the constatation that a microscopic wavy behaviour of the metric can produce a macroscopic energy tensor density like the one occurring in a material medium and simultaneously account for the macroscopic electromagnetic properties of that medium, provided that it is nonconducting and nondispersive. In order to reproduce the conducting, absorptive and dispersive properties of real media, as well as for obtaining truly realistic energy tensor densities, thus coming significantly in contact with the enormous body of experimental knowledge concerning the structure of matter, we are in patent need of field laws ruling the microscopic behaviour of $\boldsymbol{g}^{ik}$. It can be hoped that through such laws one may succeed in providing an objective description in space and time of the elementary processes of emission and absorption of radiation as resonance phenomena between waves,[16] since the nonlinear way through which $\boldsymbol{g}^{ik}$ enters the conservation identities (9) is in principle apt to achieve this result. The modest goal of the present paper has been to outline, through particular examples, some theoretical opportunities offered by the geometrical objects introduced long ago by Einstein and by Schrödinger; these opportunities may become of interest if the conceptual and technical difficulties that the program of quantization encounters when confronted with the dynamic structure of spacetime do not find a satisfactory solution.

Appendix I

Since the metrics s_{ik} and $\bar{s}_{ik}$ are conformally related:

$$\Sigma^i_{kl}(s_{ab})=\Sigma^i_{kl}(\bar{s}_{ab})+(1/2)(\delta_k{}^i\sigma_{,l}+\delta_l{}^i\sigma_{,k}-\bar{s}^{im}\bar{s}_{kl}\sigma_{,m}), \tag{A·1}$$

$$S_{ik}(s_{ab})=S_{ik}(\bar{s}_{ab})-\sigma_{,i|k}+(1/2)[\sigma_{,i}\sigma_{,k}-\bar{s}_{ik}\bar{s}^{am}(\sigma_{,a|m}+\sigma_{,a}\sigma_{,m})], \tag{A·2}$$

$$S(s_{ab})=e^{-\sigma}[S(\bar{s}_{ab})-3\bar{s}^{pq}(\sigma_{,p|q}+(1/2)\sigma_{,p}\sigma_{,q})]. \tag{A·3}$$

We get

$$\langle j_{i,k}-j_{k,i}\rangle=\bar{j}_{i,k}-\bar{j}_{k,i} \tag{A·4}$$

and

$$\langle S(s_{ab})a_{ik}\rangle=S(\bar{s}_{ab})\bar{a}_{ik}-(9/2)\bar{a}_{ik}\bar{s}^{pq}c_{pq}. \tag{A·5}$$

In order to evaluate $\langle a_{ik}\vdots^a_a\rangle$ we note that

$$a_{ik}\vdots^a_a=(-\bar{s})^{-1/2}\bar{s}_{ip}\bar{s}_{kq}s^{rs}\boldsymbol{a}^{pq}{}_{;r;s}. \tag{A·6}$$

Due to (A·1) we write

$$\begin{aligned}\boldsymbol{a}^{pq}{}_{;r}=&\boldsymbol{a}^{pq}{}_{|r}-\boldsymbol{a}^{pq}\sigma_{,r}+(1/2)(\boldsymbol{a}^{nq}\delta_r{}^p-\boldsymbol{a}^{np}\delta_r{}^q)\sigma_{,n}\\ &-(1/2)(\boldsymbol{a}^{nq}\bar{s}^{pm}-\boldsymbol{a}^{np}\bar{s}^{qm})\bar{s}_{nr}\sigma_{,m}\end{aligned} \tag{A·7}$$

and we find

$$\langle s^{rs}\boldsymbol{a}^{pq}{}_{;r;s}\rangle=\boldsymbol{a}^{pq}{}_{|a}^{|a}+\langle\boldsymbol{a}^{pq}{}_{;r}\bar{s}^{rs}\sigma_{,s}+(1/2)(\boldsymbol{a}^{tq}{}_{;r}\bar{s}^{rp}-\boldsymbol{a}^{tp}{}_{;r}\bar{s}^{rq})\sigma_{,t}\rangle, \tag{A·8}$$

where

$$\langle\boldsymbol{a}^{pq}{}_{;r}\bar{s}^{rs}\sigma_{,s}\rangle=-\boldsymbol{a}^{pq}\bar{s}^{rs}c_{rs} \tag{A·9}$$

and

$$\langle\boldsymbol{a}^{tq}{}_{;r}\bar{s}^{rp}\sigma_{,t}\rangle=(1/2)(\boldsymbol{a}^{pt}\bar{s}^{rq}-\boldsymbol{a}^{tq}\bar{s}^{rp}-\boldsymbol{a}^{pq}\bar{s}^{rt})c_{rt}. \tag{A·10}$$

The overall result for $\bar{B}_{[ik]}$ is

$$\begin{aligned}\bar{B}_{[ik]}=&(2\pi/3)(\bar{j}_{i,k}-\bar{j}_{k,i})+(1/2)[\bar{a}^{pq}C_{pqik}(\bar{s}_{ab})+(1/3)S(\bar{s}_{ab})\bar{a}_{ik}+\bar{a}_{ik}{}_{|a}^{|a}]\\ &-(3/2)\bar{s}^{pq}c_{pq}\bar{a}_{ik}.\end{aligned} \tag{A·11}$$

Appendix II

In this case the nonvanishing components of Σ^i_{kl} read

$$\begin{aligned}&\Sigma^\lambda_{\mu\nu}=(1/2)(\delta_\mu{}^\lambda\sigma_{,\nu}+\delta_\nu{}^\lambda\sigma_{,\mu}-\eta^{\lambda\rho}\eta_{\mu\nu}\sigma_{,\rho}),\\ &\Sigma^4_{\mu\nu}=-(1/2)\eta_{\mu\nu}\eta^{44}e^\sigma\sigma_{,4},\quad \Sigma^\lambda_{4\nu}=(1/2)\delta_\nu{}^\lambda\sigma_{,4};\end{aligned} \tag{A·12}$$

Greek indices run from 1 to 3. The components of S_{ik} are

$$\begin{aligned}S_{\lambda\mu}=&-(1/2)[\sigma_{,\lambda,\mu}-(1/2)\sigma_{,\lambda}\sigma_{,\mu}+\eta_{\lambda\mu}\eta^{\alpha\beta}(\sigma_{,\alpha,\beta}+(1/2)\sigma_{,\alpha}\sigma_{,\beta})\\ &+\eta_{\lambda\mu}\eta^{44}e^\sigma(\sigma_{,4,4}+(3/2)\sigma_{,4}\sigma_{,4})],\\ S_{4\mu}=&-\sigma_{,4,\mu},\quad S_{44}=-(3/2)(\sigma_{,4,4}+(1/2)\sigma_{,4}\sigma_{,4}),\end{aligned} \tag{A·13}$$

while the nonvanishing components of S_{pqik} read

$$S_{\lambda\mu\rho\tau}=(1/2)e^{\sigma}[\eta_{\lambda\rho}(\sigma_{,\mu,\tau}-(1/2)\sigma_{,\mu}\sigma_{,\tau})-\eta_{\mu\rho}(\sigma_{,\lambda,\tau}-(1/2)\sigma_{,\lambda}\sigma_{,\tau})$$
$$-\eta_{\lambda\tau}(\sigma_{,\mu,\rho}-(1/2)\sigma_{,\mu}\sigma_{,\rho})+\eta_{\mu\tau}(\sigma_{,\lambda,\rho}-(1/2)\sigma_{,\lambda}\sigma_{,\rho})]$$
$$+(1/4)(\eta_{\lambda\rho}\eta_{\mu\tau}-\eta_{\lambda\tau}\eta_{\mu\rho})(\eta^{\alpha\beta}e^{\sigma}\sigma_{,\alpha}\sigma_{,\beta}+\eta^{44}e^{2\sigma}\sigma_{,4}\sigma_{,4}),$$
$$S_{4\mu\rho\tau}=(1/2)e^{\sigma}(\eta_{\mu\tau}\sigma_{,\rho,4}-\eta_{\mu\rho}\sigma_{,\tau,4}),$$
$$S_{4\mu4\tau}=(1/2)\eta_{\mu\tau}e^{\sigma}(\sigma_{,4,4}+(1/2)\sigma_{,4}\sigma_{,4}). \tag{A·14}$$

Equation (A·4) holds again. Let us evaluate

$$\langle a_i{}^n S_{nk}\rangle=\boldsymbol{a}^{pn}\langle e^{-3\sigma/2}s_{ip}S_{nk}\rangle; \tag{A·15}$$

its components of use in the sequel are

$$\langle a_\lambda{}^n S_{n\mu}\rangle=-(1/2)(\bar{a}_{\lambda\mu}\eta^{pq}c_{pq}+\bar{a}_\lambda{}^4 c_{4\mu}),$$
$$\langle a_4{}^n S_{n\mu}\rangle=-\bar{a}_{4\mu}\eta^{pq}c_{pq}-(1/2)\bar{a}_4{}^{\varepsilon}c_{\mu\varepsilon},$$
$$\langle a_\mu{}^n S_{n4}\rangle=(3/2)\bar{a}_{4\mu}\eta^{44}c_{44}-(1/2)\bar{a}_\mu{}^{\varepsilon}c_{4\varepsilon}. \tag{A·16}$$

The components of $\langle a^{pq}S_{pqik}\rangle$ read

$$\langle a^{pq}S_{pq\lambda\mu}\rangle=(1/2)(\bar{a}_{\lambda\mu}\eta^{pq}c_{pq}+\bar{a}_\lambda{}^4 c_{4\mu}-\bar{a}_\mu{}^4 c_{4\lambda}),$$
$$\langle a^{pq}S_{pq4\mu}\rangle=-(1/2)\bar{a}_\mu{}^{\varepsilon}c_{4\varepsilon}+\bar{a}_{4\mu}\eta^{44}c_{44}. \tag{A·17}$$

In order to calculate $\langle a_{ik}{}^{\,a}_{\,;\,a}\rangle$ we first find the expression of $\boldsymbol{a}^{pq}{}_{;r}$, whose components read

$$\boldsymbol{a}^{\alpha\beta}{}_{;\gamma}=\boldsymbol{a}^{\alpha\beta}{}_{,\gamma}+(1/2)[\boldsymbol{a}^{\alpha\varepsilon}\delta_\gamma{}^\beta-\boldsymbol{a}^{\beta\varepsilon}\delta_\gamma{}^\alpha-\eta_{\gamma\delta}(\boldsymbol{a}^{\alpha\delta}\eta^{\beta\varepsilon}-\boldsymbol{a}^{\beta\delta}\eta^{\alpha\varepsilon})\sigma_{,\varepsilon}-\boldsymbol{a}^{\alpha\beta}\sigma_{,\gamma}$$
$$+(\boldsymbol{a}^{\alpha4}\delta_\gamma{}^\beta-a^{\beta4}\delta_\gamma{}^\alpha)\sigma_{,4}],$$
$$\boldsymbol{a}^{\alpha\beta}{}_{;4}=\boldsymbol{a}^{\alpha\beta}{}_{,4}-(1/2)\boldsymbol{a}^{\alpha\beta}\sigma_{,4},$$
$$\boldsymbol{a}^{4\beta}{}_{;\gamma}=\boldsymbol{a}^{4\beta}{}_{,\gamma}-\boldsymbol{a}^{4\beta}\sigma_{,\gamma}+(1/2)[(\boldsymbol{a}^{4\varepsilon}\delta_\gamma{}^\beta-\boldsymbol{a}^{4\delta}\eta_{\gamma\delta}\eta^{\beta\varepsilon})\sigma_{,\varepsilon}+\boldsymbol{a}^{\beta\delta}\eta_{\gamma\delta}\eta^{44}e^{\sigma}\sigma_{,4}],$$
$$\boldsymbol{a}^{4\beta}{}_{;4}=\boldsymbol{a}^{4\beta}{}_{,4}-\boldsymbol{a}^{4\beta}\sigma_{,4}. \tag{A·18}$$

Since

$$a_{ik}{}^{\,a}_{\,;\,a}=(-s)^{-1/2}s_{ip}s_{kq}s^{rs}\boldsymbol{a}^{pq}{}_{;r;s}, \tag{A·19}$$

we write

$$a_{\lambda\mu}{}^{\,a}_{\,;\,a}=\eta_{\lambda\alpha}\eta_{\mu\beta}(\eta^{\gamma\delta}e^{-\sigma/2}\boldsymbol{a}^{\alpha\beta}{}_{;\gamma;\delta}+\eta^{44}e^{\sigma/2}\boldsymbol{a}^{\alpha\beta}{}_{;4;4}),$$
$$a_{4\mu}{}^{\,a}_{\,;\,a}=\eta_{44}\eta_{\mu\beta}(\eta^{\gamma\delta}e^{-3\sigma/2}\boldsymbol{a}^{4\beta}{}_{;\gamma;\delta}+\eta^{44}e^{-\sigma/2}\boldsymbol{a}^{4\beta}{}_{;4;4}). \tag{A·20}$$

We find, after a straightforward calculation

$$\langle a_{\lambda\mu}{}^{\,a}_{\,;\,a}\rangle=\bar{a}_{\lambda\mu;a}{}^{a}-(1/4)[3\bar{a}_{\lambda\mu}\eta^{pq}c_{pq}-\bar{a}_\lambda{}^{\varepsilon}c_{\mu\varepsilon}+\bar{a}_\mu{}^{\varepsilon}c_{\lambda\varepsilon}+\bar{a}_\lambda{}^4 c_{\mu4}-\bar{a}_\mu{}^4 c_{\lambda4}],$$
$$\langle a_{4\mu}{}^{\,a}_{\,;\,a}\rangle=\bar{a}_{4\mu;a}{}^{a}-(1/4)[\bar{a}_{4\mu}(5\eta^{\gamma\delta}c_{\gamma\delta}+6\eta^{44}c_{44})+\bar{a}_4{}^{\varepsilon}c_{\mu\varepsilon}-\bar{a}_\mu{}^{\varepsilon}c_{4\varepsilon}], \tag{A·21}$$

where $\bar{a}_{ik;a}^{\ ;a} = \eta^{rs} \bar{a}_{ik,r,s}$.

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