Microscopic Fields and Macroscopic Averages in Einstein's Unified Field Theory*

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ABSTRACT. The problem of the relation between microscopic and macroscopic reality in the generally covariant theories is first considered, and it is argued that a sensible definition of the macroscopic averages imposes a restriction of the allowed transformations of coordinates to suitably defined macroscopic transformations. Spacetime averages of the geometric objects of a generally covariant theory are then defined, and the reconstruction of some features of macroscopic reality from hypothetic microscopic structures through such averages is attempted in the case of the geometric objects of Einstein's unified field theory. It is shown with particular examples how a fluctuating microscopic structure of the metric field can rule the constitutive relation for macroscopic electromagnetism both in vacuo and in nondispersive material media. Moreover, if both the metric and the skew field $\mathbf{a}^{ik}$ that represents the electric displacement and the magnetic field are assumed to possess a wavy microscopic behaviour, nonvanishing average generalized force densities $< \mathbf{T}^m_{k;m} >$ are found to occur in the continuum, that originate from a resonance process, in which at least three waves need to be involved. The previously required limitation of covariance to the macroscopic transformations ensures meaning to the notion of a periodic microscopic disturbance, for which a wave four-vector can be defined. Let k^A_m and k^B_m represent the wave four-vectors of two plane wave disturbances displayed by a^{ik}, while k^C_m is the wave four-vector for a plane wave perturbation of the metric; it is found that $< \mathbf{T}^m_{k;m} >$ can be nonvanishing only if the three-wave resonance condition $k^A_m \pm k^B_m \pm k^C_m = 0$, so ubiquitous in quantum physics, is satisfied. A particular example of resonant process is provided, in which $< \mathbf{T}^m_{k;m} >$ is actually nonvanishing. The wavy behaviour of the metric is essential for the occurrence of this resonance phenomenon.

RÉSUMÉ. *On examine d'abord le problème de la relation entre la réalité microscopique et la réalité macroscopique dans les théories covariantes générales, et il est montré qu'une bonne définition des moyennes macroscopiques impose une restriction aux transformations de coordonnées permises pour le cas macroscopique. On définit ensuite les moyennes dans l'espace-temps des objects géométriques d'une théorie covariante générale. La reconstitution de certaines propriétés de la*

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réalité macroscopique à partir de structures microscopiques supposées est ensuite tentée dans le cas des objects géométriques de la théorie du champ unifié d'Einstein. Il est montré par des exemples, comment une structure microscopique fluctuante du champ de la métrique peut régir la relation constitutive de l'électromagnétisme macroscopique, dans le vide comme dans des milieux matériels non-dispersifs. De plus, si la métrique, et le champ antisymétrique $\mathbf{a}^{ik}$*, qui représente le déplacement électrique et le champ magnétique, sont supposés avoir un comportement microscopique oscillant, les densités de force généralisées, non-nulles en moyenne,* $<\mathbf{T}^m_{k;m}>$ *révelént leur existence dans le milieux continu, et elles viennent d'un processus de résonance impliquant trois ondes. La limite de la covariance, nécessaire pour les transformations macroscopiques, assure une signification à la notion de perturbation périodique microscopique, pour laquelle on peut définir un quadrivecteur d'onde. On désigne par* k^A_m *et* k^B_m *les quadrivecteurs d'onde de deux perturbations sous forme d'ondes planes représentées par* a^{ik}*, tandis que* k^C_m *représente le quadrivecteur d'onde pour une onde plane perturbant la métrique; on trouve que* $<\mathbf{T}^m_{k;m}>$ *ne peut être non-nulle que si la condition de résonance des trois ondes* $k^A_m \pm k^B_m \pm k^C_m = 0$*, si omniprésente en théorie quantique, est satisfaite. Un cas particulier de processus résonant est présenté dans lequel* $<\mathbf{T}^m_{k;m}>$ *est effectivement non-nulle. L'existence de ce phénomene de résonance repose essentiellement sur le comportement oscillant de la métrique.*

1. INTRODUCTION: SOME REMARKS ON THE RELATION BETWEEN MACRO AND MICROPHYSICS IN A GENERALLY COVARIANT THEORY

The issue of the relation between macroscopic and microscopic reality as viewed through the evolution of the physical theories is a quite complex, curious problem, whose attempted solutions seem to reflect more the idiosyncrasies of the inquiring mind than an actual structure of the world. Since the direct perception of a microscopic reality is per se beyond the capability of our unassisted senses, we could have dispensed ourselves altogether with developing theories about such a hypothetical entity, had we not perceived the existence of certain macroscopic structures or processes, whose explanation looked possible through the hypothesis of some chain of causes and effects, or of some cooperative process, that related the macroscopic phenomenon displaying these structures or processes to underlying microscopic occurrences; since the regularities of these macroscopic phenomena were not dissimilar from the ones present in other situations, when no hint of a microscopic substructure was apparent, they also seemed amenable to a rational understanding, and we were forced to give up the comfortable ideal of a macroscopic physical theory closed in itself.

The natural philosopher confronted henceforth a difficult trial and error game, initially played along the following, somewhat circuitous route: he aimed at describing macroscopic reality in all its occurrences, but his mind could confidently avail only of the concepts and of the laws belonging to that sort of macroscopic theories that were just silent about the macroscopic occurrences whose explanation appealed to a microscopic substructure. As a first move, he then selected among his macroscopic concepts and laws the ones that, for some faith in the uniformity and in the simplicity of the world, he felt inclined to believe valid also at a microscopic scale, and used them for describing the behaviour of hypothetical microscopic structures, again imagined as simple, idealized replicas of some objects of macroscopic experience. The route back to macroscopic reality was then attempted via statistical hypotheses and methods.

Of course, if he fails to produce in this way a better theory of macroscopic reality with respect to the ones from which he drew inspiration, the natural philosopher can point an accusatory finger in several directions, for either the concepts and the laws that he has chosen to transfer to a small scale, or the microscopic structures that he has imagined, or else the statistical methods that he has availed upon may represent or include faulty assumptions.

An unusual amount of creativity is required at this point in order to divine what coordinated changes of the chosen concepts and laws, what invention of new microscopic structures, what new assumptions about the statistical behaviour, possibly without counterpart in the macroscopic experience, may result in a less unsatisfactory reconstruction of the macroscopic world. Hopeless as it may seem, this approach has led from Newtonian dynamics and Maxwell's theory through the electron theory of Lorentz to the Planck-Einstein-Bohr theory, then to matrix mechanics, to Schrödinger's equation, to Dirac's equation and to quantum electrodynamics. In retrospect, it is surprising how much heuristic value was already contained in the starting point chosen by Lorentz [1], how many qualitative features of macroscopic reality could already be accounted for by simply transferring to a small scale the knowledge gathered about macroscopic dynamics and macroscopic electromagnetism in vacuo. The heuristic value of Lorentz' attempt did not vanish even after its failure was ascertained, but persisted under several respects also through the subsequent developments; quantum mechanics and quantum electrodynamics may be viewed as the outcome of the efforts aimed at understanding what changes in the concepts, in the laws and in the statistical assumptions needed to be introduced in order to lead to completion Lorentz' program without renouncing two of its basic tenets: the adoption of Maxwell's electromagnetism in vacuo as a formal ingredient relevant at a microscopic scale, and the associated concept of the charged point particle.

The whole transition from the electron theory of Lorentz to quantum electrodynamics occurred by retaining the inertial reference frame as the appropriate spacetime background for the physical processes; general relativity, which appeared during that transition, had no rôle in it: the empirical confirmation achieved by general relativity in correcting certain small discrepancies between Newton's gravitodynamics and the astronomical observations led to view this theory of Einstein in a purely macroscopic perspective, and its essential novelties, like the abandonment of the inertial frame and its unique interplay between matter and spacetime structure appeared, apart from notable exceptions [2,3], as useless complications in the difficult task of providing, through a careful formulation of hypotheses of a microscopic character, a precise account of the manifold aspects of matter and of radiation. The problem of the relation between macro and microphysics in the generally covariant theories started to attract considerable attention only when quantum theory had evolved, in the minds of the theoreticians, from the instrumental condition of a set of hypotheses well suited to overcome the failure of the electrodynamic program initiated by Lorentz to the status of a general system of axioms, prescribing the formal framework within which any field theory must be inscribed in order to become properly "microscopic" and hopefully entitled to provide a physically correct answer once the way back to the macroscopic scale is suitably completed through statistical methods. It was then felt that general relativity, as the best available theory of the gravitational field, had to be quantized: the principle of uniformity in the description of the physical world seemed to impose this task.

If, confronted with the issue of quantizing general relativity, and with the long sequence of conceptual and of technical impasses that this attempt has encountered since its inception, we look back for inspiration at the path that has led from the original program of Lorentz to quantum electrodynamics, we note that each new step along that path has been taken in order to overcome some defect or failure in the description of the experimental facts by the theoretical model achieved in the previous step, while in the case of gravitation one notes a disconcerting lack of constraining experimental evidence comparable to, say, the existence of the Balmer lines or of the blackbody radiation spectrum, that would provide guidance and dispel the dangers of academicism from a theoretical effort otherwise motivated by essentially formal reasons.

General relativity, however, is not just a field theory for macroscopic gravitation; it looks rather like the first, provisional achievement of a program aimed at representing the whole of physical reality in a new way that dispenses with the need of the inertial reference frame and posits a direct relation between spacetime structure and material properties; due to these essential novelties, common to all the generally covariant theories, one should be prepared to acknowledge that for these theories the issue of the relation between macro and microphysics may well require a totally different approach from the one successfully adopted with the theories that retain the inertial frame; it may be more appropriate then to draw free inspiration from the historical sequence of attempts that has led from the electron theory by Lorentz to quantum mechanics, rather than stick to the formal expression of the end results of that endeavour, that was rooted in so different a conceptual framework.

According to this spirit one could try, as a first attempt, to transfer the driving ideas of Lorentz' program in the new environment, i.e. one should select concepts and laws from the available generally covariant theories and tentatively extrapolate them to a small scale; one should then invent microscopic structures built up with the geometric objects [4] of these theories and try the way back to macroscopic reality via statistical assumptions and methods. Due to the nonlinearity of the generally covariant theories, totally new possibilities will appear along the back and forth route between micro and macrophysics, as it was already intimated, in the framework of the Riemannian geometry, by the investigations performed by C. Lanczos [5,6]. These new possibilities are by no means confined to the realm of gravitational physics; we shall not fear the risk of academicism, since the whole of the experimental knowledge gathered about the structure and the behaviour of matter and of radiation will be in principle at our disposal for testing the validity of concepts, structures and laws that we may propose.

2. MACROSCOPIC AVERAGES IN THE GENERALLY COVARIANT THEORIES

In a generally covariant framework the very definition of macroscopic averages deserves a close scrutiny, as it is intimated e.g. by the investigations [7] dealing with such averages in cosmology. Here we wish to point out that the definition of a macroscopic average appears related to general covariance in a peculiar way. Imagine that one chooses a given generally covariant theory and tentatively assumes that its geometric objects and its laws are meaningful at any scale; their mathematical expression permits this hypothesis. The assumed general covariance of the theory will allow for general transformations of coordinates $x'^i = f^i(x^k)$ in which the functions f^i need only to satisfy appropriate conditions of continuity, differentiability, and regularity of the functional determinant $det(\partial x'^i/\partial x^k)$, but are otherwise arbitrary. For instance $f^i(x^k)$ can display a microscopic structure; this possibility is consistent with the behaviour of a geometric object $O^{mn..}_{ik..}(x^p)$ at a microscopic scale. In the following we shall write simply $O(x^p)$ for the generic geometric object whenever this shorthand does not cause confusion. Assume now that the way back from micro to macrophysics entails some averaging process, performed in the coordinate system x^i, through which some average quantity[1] $\bar{O}(x^p)$ is extracted from the unaveraged one $O(x^p)$ by some mathematical procedure, intended to mimic a process of measurement performed through a macroscopic device in the reference frame associated with the coordinate system x^i. In compliance with our ideas about averages and macroscopic reality, we expect that in the average field $\bar{O}(x^p)$ all the microscopic structures displayed by $O(x^p)$ are completely effaced, i.e., that we can associate to a generic point x^i_0 a box containing the points for which

$$|x^i - x^i_0| < \beta^i, \tag{1}$$

where the positive numbers β^i are exceedingly large with respect to the increments $|x^i - x^i_0|$ over which a variation of $O(x^p)$ becomes perceptible, and yet exceedingly small with respect to the increments $|x^i - x^i_0|$ over which a variation of $\bar{O}(x^p)$ can be appreciated.

In what manner shall $\bar{O}(x^p)$ behave under a coordinate transformation? It seems desirable that the quantity $\bar{O}(x^p)$ replicate the transformation properties of $O(x^p)$ but, if we insist that the averaged field shall transform like the unaveraged one under all the admissible transformations of coordinates [8], our expectation about the effacement of the microscopic structures cannot be realized. In fact, let us assume for instance that the components of $\bar{O}(x^p)$ display a complete cancellation of the microscopic structure; a transformation of coordinates $x^i = f^i(x^k)$ exhibiting some microscopic vagary will suffice in reintroducing the unwanted microscopic structure in the components $\bar{O}'^{mn..}_{ik..}(x^p)$ of the average field, defined with respect to the primed coordinate system.

The effacing ability of the averaging procedure is compatible with the requirement that $\bar{O}(x^p)$ transform according to the same rule obeyed by the geometric object $O(x^p)$

[1] Unlike the expression "geometric object", here and in the following, the word "quantity" is not used in the strict technical sense of Ref. 4.

only for the subset of transformations $x'^i = h^i(x^k)$ such that, if h^i and its derivatives up to some appropriate order are expanded in Taylor's series around the generic point x_0^i:

$$x'^i = h^i(x_0^k) + (h^i_{,m})_0(x^m - x_0^m) + (1/2)(h^i_{,m,n})_0(x^m - x_0^m)(x^n - x_0^n) + ..., \tag{2}$$

$$x'^i_{,m} = (h^i_{,m})_0 + (h^i_{,m,n})_0(x^n - x_0^n) + ..., \tag{3}$$

and so on, the leading term in each expansion is exceedingly larger than the subsequent ones for all the points x^i within the box defined by (1). In order to retain the distinction between a microscopic and a macroscopic scale we shall admit only coordinate transformations that fulfil this condition[2]; they will be called henceforth macroscopic, and also macroscopic will be called all the coordinate systems that can be reached from the coordinate system x^i through a macroscopic transformation, provided that the average quantities defined in the system x^i display the required effacement of the microscopic structures.

Up to now no word has been said about the definition of average that we intend to adopt; the necessary restriction of covariance to the previously defined macroscopic transformations eases the problem, since the very form in which this restriction was expressed suggests the following simple scheme of spacetime averaging [9] as an admissible choice. Let us begin by introducing in the spacetime region where the averages must be defined a coordinate system x^i, and by associating to each point x^i a neighbourhood $\Omega(x^i)$ according to the following prescription: we surround a given point x_0^i with a box defined by (1), that we choose as $\Omega(x_0^i)$; the neighbourhood $\Omega(x_0^i + \delta x^i)$ associated with the point whose coordinates are $x_0^i + \delta x^i$ is then simply defined as the box for which

$$|x^i - x_0^i - \delta x^i| < \beta^i; \tag{4}$$

in this way a neighbourhood is associated to each point in the spacetime region under question. This association will be kept in all the allowed coordinate systems, i.e. if x'^i and x^i denote the same point in two coordinate systems, $\Omega(x'^i)$ shall contain the same points as $\Omega(x^i)$. The spacetime average of the field $O(x^p)$ is then defined as

$$\bar{O}(x^p) \equiv < O >_{\Omega(x^p)} = \frac{\int_{\Omega(x^p)} O d\Omega}{\int_{\Omega(x^p)} d\Omega}; \tag{5}$$

[2] The idea that a restriction of covariance is needed in order to establish a distinction between a macroscopic and a microscopic scale is present in the literature that deals with the problem of defining averages in cosmology. See e.g. the Introduction of the paper by A. H. Nelson (Ref. 7), where such a restriction is invoked as a necessary means of discriminating between the global and the local properties of the metric, and the paper by T. W. Noonan (Ref. 9). According to the latter author we must postulate a duality, i.e. the existence of two types of observers, a macroscopic observer who can see only the large-scale properties of the medium, and a microscopic observer who can see the small-scale properties. As regards the allowed coordinate transformations, it is the macroscopic observer who, according to Noonan, imposes the greater constraint, since he is by definition unable to perceive coordinate transformations endowed with a microscopic structure.

it is a field that, with adequate accuracy, transforms under the macroscopic coordinate transformations according to the same law as the geometric object $O(x^p)$; moreover, in the coordinate system x^i one finds exactly

$$< O_{,r} >_{\Omega(x^p)} = \bar{O}_{,r}(x^p), \tag{6}$$

and the same property holds with adequate accuracy in all the macroscopic coordinate systems.

3. THE GEOMETRIC OBJECTS OF THE UNIFIED FIELD THEORIES OF EINSTEIN AND SCHRÖDINGER

We have stressed that the nonlinearity of the generally covariant theories offers completely new possibilities in the back and forth game of reconstructing the macroscopic reality from hypothetical microscopic structures and laws; this paper aims at evidencing two such possibilities offered through the geometric objects of the non-Riemannian theories developed by Einstein and by Schrödinger in their search [10,11] for an extension of the general relativity of 1915 that could encompass both gravitation and electromagnetism. In retrospect, one does not see really cogent reasons why these theories should provide, as hoped for by their authors, field theoretical completions of general relativity: their geometric objects are so closely akin to the ones occurring in that theory, that one may well wonder [12] why in such theories one should depart from the attitude kept in general relativity, where one has not to do with field laws describing the evolution of matter, but rather with a fundamental definition of the stress-momentum-energy tensor in terms of the metric [13,14]. It seems reasonable to assume that a similar situation should prevail also in the above mentioned field theories, and to investigate what new definitions of physical quantities can be given through the geometric objects first envisaged by Einstein.

One possible identification [15] of those geometric objects with physical entities runs as follows: in a four-dimensional manifold endowed with real coordinates x^i a nonsymmetric tensor density $\mathbf{g}^{ik}$ defines [16] through its symmetric part $\mathbf{g}^{(ik)}$ the metric tensor s_{ik}:

$$\mathbf{s}^{ik} = \mathbf{g}^{(ik)}, \quad \mathbf{s}^{ik} = (-s)^{1/2} s^{ik}, \quad s^{im} s_{km} = \delta^i_k, \quad s = det(s_{ik}), \tag{7}$$

while its skew part $\mathbf{g}^{[ik]} \equiv \mathbf{a}^{ik}$ defines the electric displacement $\mathbf{D}$ and the magnetic field $\mathbf{H}$ through the identifications:

$$(\mathbf{a}^{41}, \mathbf{a}^{42}, \mathbf{a}^{43}) \Rightarrow (D_1, D_2, D_3), \quad (\mathbf{a}^{23}, \mathbf{a}^{31}, \mathbf{a}^{12}) \Rightarrow (H_1, H_2, H_3); \tag{8}$$

the electric four-current density $\mathbf{j}^i$ is correspondingly defined as

$$\mathbf{j}^i = (1/4\pi) \mathbf{g}^{[is]}{}_{,s}. \tag{9}$$

Through the equation

$$\mathbf{g}^{qr}{}_{,p} + \mathbf{g}^{sr}\Gamma^{q}_{sp} + \mathbf{g}^{qs}\Gamma^{r}_{ps} - \mathbf{g}^{qr}\Gamma^{t}_{(pt)} = (4\pi/3)(\mathbf{j}^{q}\delta^{r}_{p} - \mathbf{j}^{r}\delta^{q}_{p}) \tag{10}$$

the tensor density $\mathbf{g}^{ik}$ uniquely [17,18] determines the nonsymmetric affine connection Γ^{i}_{km}, by definition constrained to yield $\Gamma^{k}_{[ik]} = 0$, through which the symmetrized Ricci tensor

$$B_{ik}(\Gamma) = \Gamma^{a}_{ik,a} - (1/2)(\Gamma^{a}_{ia,k} + \Gamma^{a}_{ka,i}) - \Gamma^{a}_{ib}\Gamma^{b}_{ak} + \Gamma^{a}_{ik}\Gamma^{b}_{ab} \tag{11}$$

is constructed [19]. The reason why this symmetrized tensor is considered in place of the plain one occurring in general relativity will soon be apparent. The symmetric part of B_{ik} is assumed to define the symmetric stress-momentum-energy tensor T_{ik} of a material medium through the equation

$$B_{(ik)}(\Gamma) = 8\pi(T_{ik} - (1/2)s_{ik}s^{pq}T_{pq}), \tag{12}$$

while its skew part $B_{[ik]}$ is identified with the electric field $\mathbf{E}$ and with the magnetic induction $\mathbf{B}$ through the rule

$$(B_{[14]}, B_{[24]}, B_{[34]}) \Rightarrow (E_1, E_2, E_3), \quad (B_{[23]}, B_{[31]}, B_{[12]}) \Rightarrow (B_1, B_2, B_3); \tag{13}$$

the magnetic four-current K_{ikm} is consequently defined as

$$K_{ikm} = (3/8\pi)B_{[[ik],m]}, \tag{14}$$

where $B_{[[ik],m]} \equiv (1/3)(B_{[ik],m} + B_{[km],i} + B_{[mi],k})$. Thanks to equation (10) and to the definition (11) it happens that, if T_{ik}, $\mathbf{j}^{i}$ and K_{ikm} are the material counterpart of a given field $\mathbf{g}^{ik}$, the matter counterpart of the transposed field $\tilde{\mathbf{g}}^{ik} \equiv \mathbf{g}^{ki}$, that we indicate with $\tilde{T}_{ik}$, $\tilde{\mathbf{j}}^{i}$ and $\tilde{K}_{ikm}$, is such that $\tilde{T}_{ik} = T_{ik}$, $\tilde{\mathbf{j}}^{i} = -\mathbf{j}^{i}$ and $\tilde{K}_{ikm} = -K_{ikm}$, i.e. "the requirement that positive and negative electricity enter symmetrically into the laws of physics" [20] is satisfied. When $\mathbf{j}^{i}$ is not vanishing, this requirement cannot be fulfilled if, instead of B_{ik}, the plain Ricci tensor is adopted.

The consistency of the identifications introduced above appears from the contracted Bianchi identities, that can be written [15] as

$$\mathbf{T}^{m}_{k;m} = (1/2)(\mathbf{j}^{i}B_{[ki]} + K_{ikm}\mathbf{g}^{[mi]}), \tag{15}$$

where $\mathbf{T}^{m}_{k} = \mathbf{s}^{mi}T_{ki}$, and ";" indicates the covariant differentiation performed with the Christoffel affine connection

$$\Sigma^{i}_{km} = (1/2)s^{ia}(s_{ak,m} + s_{am,k} - s_{km,a}) \tag{16}$$

associated with the metric s_{ik} (that will be hereafter used to move indices, to build tensor densities from tensors, and vice-versa). From (15) one gathers than the local nonconservation of the energy tensor in the Riemannian spacetime defined by the metric s_{ik} is due to the Lorentz coupling of the electric four-current to $B_{[ik]}$ and of the magnetic four-current to $\mathbf{g}^{[ik]}$, as one expects to occur in the electrified material medium of a gravito-electromagnetic theory. Two versions of this theoretical structure are possible, according to whether $\mathbf{g}^{ik}$ is a real nonsymmetric, or a complex Hermitian tensor density.

4. THE CONSTITUTIVE RELATION FOR MICROSCOPIC ELECTROMAGNETISM

Assuming, as we are doing, that the sort of electromagnetism that we are reading off the geometric objects of Einstein's unified field theory is competent at a microscopic scale means a substantial departure from the letter of Lorentz' approach. In that case, a simple hypothesis is made for the relation between inductions and fields that should prevail microscopically: if the skew tensor density $\mathbf{a}^{ik}$ represents as before the electric displacement and the magnetic field, the skew tensor b_{ik} that defines the electric field and the magnetic induction is given by

$$b_{ik} = a_{ik} \equiv (-s)^{-1/2} s_{ip} s_{kq} \mathbf{a}^{pq}, \tag{17}$$

i.e. by an algebraic expression in terms of s_{ik} and of $\mathbf{a}^{ik}$, in which $\mathbf{a}^{ik}$ enters in a linear way. We have written this constitutive relation in curvilinear coordinates for contrasting it with the one that exists instead between $\mathbf{a}^{ik}$ and $B_{[ik]}(\Gamma)$, a nonlinear, differential relation which is the antisymmetric counterpart of the relation between the metric tensor density $\mathbf{s}^{ik}$ and the symmetric field $B_{(ik)}$ that defines through (12) the stress-momentum-energy content of the manifold; both these relations are simultaneously found by solving (10) for Γ^i_{km} and by substituting its expression in (11). Let

$$S^i_{kmn} = \Sigma^i_{km,n} - \Sigma^i_{kn,m} - \Sigma^i_{am}\Sigma^a_{kn} + \Sigma^i_{an}\Sigma^a_{km} \tag{18}$$

be the Riemann tensor defined with the Christoffel symbol Σ^i_{km}, and assume that $\mathbf{a}^{ik}$ is a vanishingly small quantity; the linear approximation to $B_{[ik]}$ then reads [15]

$$B_{[ik]} = (2\pi/3)(j_{i,k} - j_{k,i}) + (1/2)(a_i{}^n S_{nk} - a_k{}^n S_{ni} + a^{pq} S_{pqik} + a_{ik;a}{}^{;a}), \tag{19}$$

where $S_{ik} \equiv S^p_{ikp}$ is the Ricci tensor of s_{ik} and $a_{ik}{}^{;m} \equiv s^{mn} a_{ik;n}$. This equation shows how widely the constitutive relation for microscopic electromagnetism that we are adopting departs from the one assumed by Lorentz already for weak inductions and fields. The right-hand side of (19) is homogeneous of degree two with respect to differentiation; therefore the small scale behaviour of both $\mathbf{a}^{ik}$ and $\mathbf{s}^{ik}$ will be crucial in ruling the relation between inductions and fields, as it is fundamental in determining the stress-momentum-energy content of matter; the same assertion holds for the generalized force density felt by the electrified medium, given by (15). As a consequence, a whole new range of possibilities is offered in the game of reconstructing macroscopic reality from a hypothetic microscopic behaviour, which has no counterpart in theories in which the constitutive relation (17) is instead adopted at a microscopic scale.

5. MICROSCOPIC FLUCTUATIONS OF THE METRIC CAN RULE THE MACROSCOPIC CONSTITUTIVE RELATION IN VACUO AND IN NONDISPERSIVE MEDIA

Imagine for instance that, while $\mathbf{a}^{ik}$ is very small and varying only at a macroscopic scale, s_{ik} exhibits a microscopic structure. We can write

$$s_{ik} = \bar{s}_{ik} + \delta s_{ik}, \tag{20}$$

where $\bar{s}_{ik}$ means the average metric calculated according to the definition (5), while δs_{ik} indicates a microscopic fluctuation. We shall assume that $|\delta s_{ik}|$ is very small with respect to $|s_{ik}|$, and that $|\delta s_{ik}| \ll |\delta s_{ik,m}| \ll |\delta s_{ik,m,n}|$, since the characteristic length of the fluctuations is microscopic, i.e. quite small in our units. What is the behaviour of the average field $\bar{B}_{[ik]}$ under these conditions? We can avail of the expression (19) in order to provide a first answer, limited to the linear approximation in $\mathbf{a}^{ik}$. This expression can be expanded into a sum of addenda, each one given by $\mathbf{a}^{ik}$, or $\mathbf{a}^{ik}{}_{,m}$, or else $\mathbf{a}^{ik}{}_{,m,n}$, times a product of several terms, individually given by s_{ik}, s^{ik}, $(-s)^{-1/2}$ and by the ordinary derivatives of s_{ik} up to second order, that we call metric factor, because only the metric appears in it; in a metric factor containing $s_{ik,m,n}$ no further derivatives are allowed. Since $\mathbf{a}^{ik}$ is assumed to vary at a macroscopic scale, averaging the individual addendum reduces to calculating the mean of the corresponding metric factor. Let us turn each metric factor displaying a second derivative into the overall derivative of a metric factor in which s_{ik} is differentiated once, minus the sum of metric factors that contain the product of two first derivatives. Due to the previously made assumptions and to (6), the whole problem of averaging $B_{[ik]}$ thus reduces to evaluating the means of metric factors where only the metric and its first derivatives are present; the latter can appear at most twice in a given metric factor.

The mean of a metric factor where no derivatives appear is known, since , due to the smallness of the fluctuations, we can write

$$< s_{ik}..s^{pq}..(-s)^{-1/2} >= \bar{s}_{ik}..\bar{s}^{pq}..(-\bar{s})^{-1/2}; \tag{21}$$

we assume that the smallness of the fluctuations is so related to the shortness of their characteristic length that we can write also

$$< s_{ik}..s^{pq}s_{rs,t}..(-s)^{-1/2} >= \bar{s}_{ik}..\bar{s}^{pq}\bar{s}_{rs,t}..(-\bar{s})^{-1/2}; \tag{22}$$

while the evaluation of

$$< s_{ik}..s^{pq}s_{rs,t}s_{uv,z}..(-s)^{-1/2} >= \bar{s}_{ik}..\bar{s}^{pq} < s_{rs,t}s_{uv,z} > ..(-\bar{s})^{-1/2}; \tag{23}$$

will require hypotheses of a statistical character on the microscopic behaviour of the metric, since $< s_{ik,m}s_{np,q} >$ will differ strongly from $\bar{s}_{ik,m}\bar{s}_{np,q}$. The quantity

$$F_{ikmnpq} =< s_{ik,m}s_{np,q} > -\bar{s}_{ik,m}\bar{s}_{np,q}, \tag{24}$$

which, due to the previous assumptions, behaves as a tensor under macroscopic coordinate transformations, is the appropriate object for encoding the statistical information required for the explicit calculation of $\bar{B}_{[ik]}$.

In the particular case, when the fluctuating metric is conformally related [21] to its average, i.e. when $s_{ik} = e^{\sigma}\bar{s}_{ik}$, with $|\sigma| \ll 1$, we get

$$F_{ikmnpq} = \bar{s}_{ik}\bar{s}_{np} < e^{2\sigma}\sigma_{,m}\sigma_{,q} >= \bar{s}_{ik}\bar{s}_{np}c_{mq}, \tag{25}$$

and the statistical information is expressed by the symmetric quantity c_{ik}, that behaves as a tensor under macroscopic transformations. A calculation of $\bar{B}_{[ik]}$ under these conditions [22] leads to the result

$$< B_{[ik]}(s_{ab}, a_{ab}) >= B_{[ik]}(\bar{s}_{ab}, \bar{a}_{ab}) + D\bar{a}_{ik}, \tag{26}$$

where $D = -(3/2)\bar{s}^{pq}c_{pq}$, and the function $B_{[ik]}(s_{ab}, a_{ab})$ is given by (19). The first term at the right-hand side of (26) displays on the average fields $\bar{s}_{ik}$ and $\bar{a}_{ik}$ the same dependence that the linear approximation to $B_{[ik]}$ has on s_{ik} and a_{ik}. The second term is just given by the average of a_{ik} times a factor D that behaves as a scalar under macroscopic transformations. If D is constant in a given spacetime region and its magnitude is such that $D\bar{a}_{ik}$ is by far the dominant term at the right-hand side of (26), the usual constitutive relation (17) appropriate to the macroscopic vacuum is found to prevail between $\mathbf{a}^{ik}$ and $\bar{B}_{[ik]}$. Under these circumstances, if the mean magnetic current $\bar{K}_{ikm}$ is vanishing, as one assumes in macroscopic electromagnetism, one finds

$$< B_{[[ik],m]} >= D\bar{a}_{[[ik],m]} = 0, \tag{27}$$

i.e. the macroscopic inductions and fields fulfil the usual equations for vacuum, and the average of the right-hand side of (15) exhibits the usual force density felt in vacuo by a macroscopic electric four-current $\mathbf{j}^i$.

Although this medium has a weak-field electromagnetic behaviour that may exactly reproduce the one appropriate to the macroscopic vacuum, its material content is by no means vanishing, not either in the average sense, also when $\bar{s}_{ik}$ is a vacuum metric. Let us calculate the mean of the stress-momentum-energy density $\mathbf{T}_k^m$; since $\mathbf{a}^{ik}$ is vanishingly small, we can neglect its contribution, and write:

$$\mathbf{T}_k^m = \mathbf{T}_k^m(s_{ab}) = \mathbf{s}^{im}[S_{ik}(s_{ab}) - (1/2)s_{ik}S(s_{ab})], \tag{28}$$

where $S = s^{pq}S_{pq}$. The previous assumptions about the fluctuations of s_{ik} suffice also for calculating this average [22]; one finds

$$8\pi < \mathbf{T}_k^m >= 8\pi\mathbf{T}_k^m(\bar{s}_{ab}) + (3/2)(-\bar{s})^{1/2}[\bar{s}^{im}c_{ik} - (1/2)\delta_k^m\bar{s}^{pq}c_{pq}]; \tag{29}$$

therefore the contribution to $< \mathbf{T}_k^m >$ coming from the conformal fluctuations cannot be made to vanish unless $c_{ik} = 0$.

Fluctuations of the metric with a lesser degree of symmetry can be used to mimic the macroscopic constitutive relation in material nonconducting media. Let us consider a simple example: suppose that a macroscopic coordinate system exists, in which

$$s_{i4} = \bar{s}_{i4}, \quad s_{\lambda\mu} = e^{\sigma}\bar{s}_{\lambda\mu}, \quad |\sigma| \ll 1, \tag{30}$$

i.e. the spatial components of the metric $s_{\lambda\mu}$ perform conformal fluctuations of very small amplitude and with a very small characteristic length around their average, while the other components are smooth; Greek indices label the spatial coordinates. We assume that besides (21) also (22) still holds; due to the choice (30), the nonvanishing components of F_{ikmnpq} will be

$$F_{\alpha\beta m\gamma\delta n} = \bar{s}_{\alpha\beta}\bar{s}_{\gamma\delta} < e^{2\sigma}\sigma_{,m}\sigma_{,n} >= \bar{s}_{\alpha\beta}\bar{s}_{\gamma\delta}c_{mn}, \tag{31}$$

and the mean components of $B_{[ik]}$ in the linear approximation (19) read

$$\begin{aligned} < B_{[\lambda\mu]}(s_{ab}, a_{ab}) > &= B_{[\lambda\mu]}(\bar{s}_{ab}, \bar{a}_{ab}) \\ &+ (1/8)[\bar{a}_{\lambda}{}^{\epsilon}c_{\mu\epsilon} - \bar{a}_{\mu}{}^{\epsilon}c_{\lambda\epsilon} - \bar{a}_{\lambda}{}^{4}c_{\mu 4} + \bar{a}_{\mu}{}^{4}c_{\lambda 4} - 5\bar{a}_{\lambda\mu}\bar{s}^{pq}c_{pq}], \\ < B_{[4\mu]}(s_{ab}, a_{ab}) > &= B_{[4\mu]}(\bar{s}_{ab}, \bar{a}_{ab}) \\ &+ (1/8)[\bar{a}_{\mu}{}^{\epsilon}c_{4\epsilon} - 3\bar{a}_{4}{}^{\epsilon}c_{\mu\epsilon} - \bar{a}_{4\mu}(9\bar{s}^{\gamma\delta}c_{\gamma\delta} + 12\bar{s}^{44}c_{44})]. \end{aligned} \tag{32}$$

If the first terms at the right-hand sides are negligible with respect to the remaining ones, (32) expresses the constitutive relation for macroscopic electromagnetism in a linear, nondissipative, nondispersive medium, which is spatially anisotropic, nonreciprocal[3] and nonuniform [23], unless more specialized assumptions are made for the behaviour of s_{ik}; for instance if, in the chosen coordinate system, we have

$$\bar{s}_{ik} = \eta_{ik} \equiv diag(-1,-1,-1,1), \quad c_{\lambda\mu} = \alpha\eta_{\lambda\mu}, \quad c_{\lambda 4} = 0, \quad c_{44} = \beta, \tag{33}$$

where α and β are constants, (32) becomes

$$\begin{aligned} < B_{[\lambda\mu]}(s_{ab}, a_{ab}) > &= B_{[\lambda\mu]}(\bar{s}_{ab}, \bar{a}_{ab}) - (1/8)(13\alpha + 5\beta)\bar{a}_{\lambda\mu}, \\ < B_{[4\mu]}(s_{ab}, a_{ab}) > &= B_{[4\mu]}(\bar{s}_{ab}, \bar{a}_{ab}) - (1/8)(30\alpha + 12\beta)\bar{a}_{4\mu}, \end{aligned} \tag{34}$$

and, if the first terms at the right-hand sides are negligible with respect to the other ones, the electromagnetic medium will be uniform, isotropic and reciprocal.

When the contribution of $\mathbf{a}^{ik}$ to $\mathbf{T}_k^m$ is neglected, the average components of the stress-momentum-energy density of the anisotropic, nonreciprocal, nonuniform electromagnetic medium read:

---

[3] Let $\mathbf{a}^{ik}$ and b_{ik} have the geometric and physical meaning that was attributed to them at the beginning of Section 4. In a linear nondissipative, nondispersive medium they are related by the equation $\mathbf{a}^{ik} = (1/2)\mathbf{X}^{ikpq}b_{pq}$, where $\mathbf{X}^{ikpq}$ is the constitutive tensor density of the medium. Let $X^{ikpq} \equiv (-s)^{-1/2}\mathbf{X}^{ikpq}$ be the corresponding tensor: a medium is called reciprocal if X^{ikpq} is invariant under reversal of the time coordinate; if not, the medium is called nonreciprocal.

$$
\begin{aligned}
8\pi < \mathbf{T}^{\nu}_{\mu}(s_{ab}) > &= 8\pi \mathbf{T}^{\nu}_{\mu}(\bar{s}_{ab}) + (1/2)\bar{s}^{\lambda\mu}c_{\lambda\mu} - (1/4)\delta^{\nu}_{\mu}(\bar{s}^{\alpha\beta}c_{\alpha\beta} + 3\bar{s}^{44}c_{44}),\\
8\pi < \mathbf{T}^{4}_{\mu}(s_{ab}) > &= 8\pi \mathbf{T}^{4}_{\mu}(\bar{s}_{ab}) + (3/2)\bar{s}^{44}c_{4\mu},\\
8\pi < \mathbf{T}^{\mu}_{4}(s_{ab}) > &= 8\pi \mathbf{T}^{\mu}_{4}(\bar{s}_{ab}) + (1/2)\bar{s}^{\mu\lambda}c_{4\lambda},\\
8\pi < \mathbf{T}^{4}_{4}(s_{ab}) > &= 8\pi \mathbf{T}^{4}_{4}(\bar{s}_{ab}) - (1/4)\bar{s}^{\alpha\beta}c_{\alpha\beta} + (3/4)\bar{s}^{44}c_{44},
\end{aligned}
\tag{35}
$$

while the nonvanishing components of $< \mathbf{T}^{m}_{k} >$ for the uniform, isotropic, reciprocal specialization defined by (33) are

$$
\begin{aligned}
8\pi < \mathbf{T}^{\nu}_{\mu} > &= -(1/4)\delta^{\nu}_{\mu}(\alpha + 3\beta),\\
8\pi < \mathbf{T}^{4}_{4} > &= (3/4)(\beta - \alpha);
\end{aligned}
\tag{36}
$$

they correspond to a uniform mechanical continuum, endowed only with energy density and with an isotropic pressure. To sum up the results of this Section, we have shown through particular examples how microscopic fluctuations of the metric can produce dynamically the constitutive relation for weak inductions and fields that prevails macroscopically both in vacuo and in material nondispersive media, although the microscopic relation (19) has a completely different character. These fluctuations produce also an average stress-momentum-energy content of the continuum, which is however ineffective in ruling the macroscopic geometry of spacetime: $\mathbf{T}^{m}_{k}(s_{ab})$ and its average can have quite large components despite the fact that $\bar{s}_{ik}$ is for instance everywhere Minkowskian; therefore we find no objection at present against the supposed existence of this stress-momentum-energy content of the continuum, and of the microscopic behaviour of s_{ik} from which it finds its origin.

6. RESONANCES BETWEEN MICROSCOPIC WAVES OF $\mathbf{g}^{ik}$ CAN PRODUCE NET AVERAGE GENERALIZED FORCES IN THE MEDIUM

Suppose now that both s_{ik} and $\mathbf{a}^{ik}$ are endowed with a microscopic structure; an intriguing relation appears then between a coherent behaviour of the two fields at a microscopic scale and the macroscopic generalized forces that show up in the continuum. Let us assume for instance that, within a box Ω defined by (1) and with respect to the coordinate system x^{i}, s^{ik} and $a^{ik} \equiv (-s)^{-1/2}\mathbf{a}^{ik}$ can be written as

$$
\begin{aligned}
s^{ik} &= \eta^{ik} + b^{ik}_{A} sin(k^{A}_{m}x^{m} + \varphi^{A}),\\
a^{ik} &= c^{ik}_{A} sin(k^{A}_{m}x^{m} + \varphi^{A}),
\end{aligned}
\tag{37}
$$

where η^{ik} is the Minkowski metric, while $b^{ik}_{A} = b^{ki}_{A}$ and $c^{ik}_{A} = -c^{ki}_{A}$ have constant values and are so small that can be dealt with as first order infinitesimal quantities; the usual summation rule is extended to the upper case Latin index $A = 1,..,n$ numbering the progressive sinusoidal waves that have k^A_m as wave four-vector and φ^A as phase constant. When terms not linear in b^{ik}_A can be neglected we can write

$$s_{ik} = \eta_{ik} - b_{Aik} sin(k^A_m x^m + \varphi^A), \tag{38}$$

where the indices in the small quantities b^{ik}_A have been lowered with η_{ik}. We consider waves whose wavelengths and whose periods are exceedingly small with respect to the dimensions of the box; the restriction of covariance to the macroscopic transformations, that was found necessary for obtaining sensible macroscopic averages, ensures now that the concept of a microscopic periodic disturbance endowed with a wave four-vector and with a phase constant is a meaningful one in Ω: the trigonometric behaviour of s^{ik} and of a^{ik} defined by (37), that is destroyed in general by an arbitrary transformation of coordinates, is in fact preserved within the box Ω by a macroscopic transformation.

We are looking after the generalized forces that may appear at a macroscopic scale in the continuum due to the microscopic behaviour of $\mathbf{g}^{ik}$; the spacetime average

$$< \mathbf{T}^m_{k;m} >= \frac{\int_\Omega \mathbf{T}^m_{k;m} d\Omega}{\int_\Omega d\Omega} \tag{39}$$

of the generalized force density over the box Ω will be the appropriate quantity to consider. One notes that the contribution to the average of those addenda of $\mathbf{T}^m_{k;m}$ that can be written as an overall ordinary derivative with respect to some coordinate will be negligible, since s^{ik} and a^{ik} have the assumed periodic behaviour at a microscopic scale. By recalling the definitions (9) and (14) one can bring the conservation identity (15) to the form

$$16\pi\mathbf{T}^m_{k;m} = 2(\mathbf{g}^{[mi]}B_{[ik]})_{,m} + (\mathbf{g}^{[im]}B_{[im]})_{,k} - \mathbf{g}^{[im]}{}_{,k}B_{[im]}; \tag{40}$$

hence, whenever the globally differentiated terms provide a negligible contribution to the average, one can write

$$< \mathbf{T}^m_{k;m} >= -(1/16\pi) < \mathbf{g}^{[im]}{}_{,k}B_{[im]} >, \tag{41}$$

which immediately reveals that, under the above mentioned conditions, the k-th component of the mean generalized force density vanishes if $\mathbf{g}^{[im]}$ does not depend on the k-th coordinate.

Let us consider a quantity Q, expressed in terms of the s^{ik} and of the a^{ik} defined by (37), and homogeneous with respect to differentiation, like all the geometric objects that we are considering. If Q is differentiated once with respect to x^m, the resulting quantity will be the sum of terms each containing a number of factors k^A_m increased by one with respect to the number of such factors appearing in Q. We indicate generically with $[b]$ a quantity of the same order of magnitude as b^{ik}_A or c^{ik}_A, and with $[k]$ a quantity having the same order of magnitude as k^A_m. Then the largest term in the first derivatives of s^{ik}, of a^{ik}, of s_{ik}, of $a_{ik} = s_{ip}s_{kq}a^{pq}$ and of $\mathbf{g}^{ik}$ with respect to x^m is a quantity whose magnitude can be indicated with $[kb]$.

The term within brackets at the right-hand side of (41) is homogeneous of degree three with respect to differentiation; it will contain leading terms of the type $[k^3b^2]$, smaller terms like $[k^3b^3]$ and so on with higher powers of $[b]$. We decide for now to stop the calculation at the terms of magnitude $[k^3b^3]$, hence we need to know

$$B_{[ik]} = \Gamma^a_{[ik],a} - \Gamma^a_{[ib]}\Gamma^b_{(ak)} - \Gamma^a_{(ib)}\Gamma^b_{[ak]} + \Gamma^a_{[ik]}\Gamma^b_{(ab)} \tag{42}$$

up to terms $[k^2b^2]$. The affine connection Γ^i_{km} is defined by (10); since the largest term in $\mathbf{g}^{qr}{}_{,p}$ is a quantity of type $[kb]$, while the largest term in $\mathbf{g}^{ik}$ is a quantity of the order unity, in general the largest term of Γ^i_{km} will be of the type $[kb]$. Therefore, in order to evaluate up to $[k^2b^2]$ all the terms at the right-hand side of (42), one needs to know the $\Gamma^i_{(km)}$ and the $\Gamma^i_{[km]}$ appearing in the products up to quantities $[kb]$; $\Gamma^i_{[km]}$ in the differentiated term $\Gamma^a_{[ik],a}$ is instead required up to quantities of type $[kb^2]$.

When terms up to $[kb]$ are retained, $\Gamma^i_{(km)}$ is given by the Christoffel symbol Σ^i_{km} of (16), where one can replace s^{ik} with η^{ik} and s_{ik} with the approximate form (38). In order to determine $\Gamma^i_{[km]}$ with the required approximation, one considers those equations of (10) that are skew in the upper indices:

$$\mathbf{a}^{qr}{}_{,p} + \mathbf{a}^{sr}\Gamma^q_{(sp)} + \mathbf{a}^{qs}\Gamma^r_{(ps)} - \mathbf{a}^{qr}\Gamma^t_{(pt)} + \mathbf{s}^{sr}\Gamma^q_{[sp]} + \mathbf{s}^{qs}\Gamma^r_{[ps]} = (4\pi/3)(\mathbf{j}^q\delta^r_p - \mathbf{j}^r\delta^q_p). \tag{43}$$

Since the largest term in $\mathbf{a}^{ik}$ is of type $[b]$, while the largest term in $\mathbf{s}^{ik}$ is of order unity, we can solve (43) for $\Gamma^i_{[km]}$ up to terms $[kb^2]$ if we substitute the $\Gamma^i_{(km)}$ appearing in it with the Christoffel symbols Σ^i_{km} defined up to $[kb]$. If the exact Σ^i_{km} are instead substituted, we find through exact manipulations

$$\Gamma^i_{[km]} = (1/2)(a_k{}^i{}_{;m} - a_m{}^i{}_{;k} + a_{km}{}^{;i}) + (4\pi/3)(\delta^i_k j_m - \delta^i_m j_k). \tag{44}$$

When $\Gamma^i_{(km)}$ is replaced in (42) by Σ^i_{km}, as it is allowed, one can write

$$B_{[ik]} = \Gamma^a_{[ik];a}, \tag{45}$$

and due to (44) $B_{[ik]}$ acquires also in the present case the approximate form (19). This expression for $B_{[ik]}$ contains all the needed terms, and also negligible ones, that will be eventually discarded; in the Appendix it is given explicitly as a function of s^{ik} and of a^{ik}. The terms of magnitude like $[k^2b]$ occurring in $B_{[ik]}$ have the overall expression

$$(1/6)(\eta_{ip}a^{ps}{}_{,s,k} - \eta_{kp}a^{ps}{}_{,s,i}) + (1/2)\eta_{ip}\eta_{kn}\eta^{aq}a^{pn}{}_{,a,q}; \tag{46}$$

they all vary with the coordinates through a "sin" dependence. As regards their trigonometric behaviour, the individual addenda of $B_{[ik]}$ whose magnitude is like $[k^2b^2]$ can instead be grouped in two categories. To the first category, with "sin.sin" dependence, either belong terms displaying the product of a second derivative of s_{mn} times a^{pq}, or terms containing the product of a second derivative of a^{mn} times the part of magnitude $[b]$ of s_{pq}, defined by (38); the second category contains the remaining addenda, with "cos.cos" dependence, that display the product of a first derivative of s_{mn} times a first derivative of a^{pq}.

In the term within brackets at the right-hand side of (41) the skew tensor $B_{[im]}$, whose trigonometric behaviour has been just examined, is contracted with $\mathbf{g}^{[im]}{}_{,k}$, which contains terms like $[kb]$, that display "cos" dependence, and terms of magnitude $[kb^2]$, which have "sin.cos" dependence, since they are either a product of a^{im} times the first derivative of s_{pq}, or the product of $a^{im}{}_{,k}$ times the $[b]$ part of s_{pq}. Therefore the contraction $\mathbf{g}^{[im]}{}_{,k}B_{[im]}$ will contain terms like $[k^3b^2]$, arising from the product of a first and a second derivative of a^{pq}, hence displaying a "sin.cos" dependence, and terms of type $[k^3b^3]$, in which it will appear either the product of two sines times a cosine, or the product of three cosines.

Certain conclusions about the mean generalized force density can be drawn without explicitly calculating the right-hand side of (41). The overall expression for the terms of magnitude $[k^3b^2]$ occurring in $\mathbf{g}^{[im]}{}_{,k}B_{[im]}$ is

$$a^{im}{}_{,k}[(1/6)(\eta_{ip}a^{ps}{}_{,s,m} - \eta_{mp}a^{ps}{}_{,s,i}) + (1/2)\eta_{ip}\eta_{nm}\eta^{aq}a^{pn}{}_{,a,q}]; \tag{47}$$

after a rearrangement that puts globally differentiated terms in evidence, this expression takes the form

$$(1/3)(a_p{}^m{}_{,k}a^{ps}{}_{,s})_{,m} - (8\pi^2/3)(j_pj^p)_{,k} + (1/2)\eta^{aq}a_{pn,k}a^{pn}{}_{,a,q}, \tag{48}$$

where the indices are moved with η_{ik}. Since also the last term in (48) can be turned into a sum of globally differentiated quantities, one concludes that the terms of magnitude $[k^3b^2]$ do not contribute to $< \mathbf{T}^m_{k;m} >$; we note that in these terms only waves of a^{ik} can appear, i.e. the wavy behaviour of the metric has no rôle in them.

The terms of magnitude $[k^3b^3]$ occurring in $\mathbf{g}^{[im]}{}_{,k}B_{[im]}$ depend on the coordinates only through the trigonometric factors

$$\begin{aligned} &cos(k^A_px^p + \varphi^A)sin(k^B_qx^q + \varphi^B)sin(k^C_rx^r + \varphi^C),\\ &cos(k^A_px^p + \varphi^A)cos(k^B_qx^q + \varphi^B)cos(k^C_rx^r + \varphi^C), \end{aligned} \tag{49}$$

as already observed; a nonzero contribution to $< \mathbf{T}^m_{k;m} >$ coming from these terms can only take place if the averages of the trigonometric factors (49) are not all vanishing. From (A6) and (A7) of the Appendix one recognizes that the latter averages will vanish unless, for some choice of A, B, C, one of the following occurrences is realized:

$$k^A_m \pm k^B_m \pm k^C_m = 0 \quad and \quad \varphi^A \pm \varphi^B \pm \varphi^C \neq (n + 1/2)\pi \tag{50}$$

for all the values of m, and with integer n; it is intended that the signs in front of a phase φ^E and in front of the corresponding wave four-vector k^E_m are always chosen in the same way. One concludes that the terms of magnitude $[k^3b^3]$ at the right-hand side of (15) cannot produce a net average generalized force density unless the three-waves resonance condition (50) is satisfied for some choice of A, B and C; in this case one and just one of the three waves is necessarily contributed by the metric s_{ik}; the sign and the value of the individual trigonometric term will be decided by the combination of the phases of the three waves whose wave four-vectors fulfil the resonance condition. By specializing (37) to a particularly simple instance one can ascertain that $< \mathbf{T}^m_{k;m} >$ can actually be nonvanishing; we reach the conclusion that the geometric objects of Einstein's unified field theory may be used to represent the production of a macroscopic generalized force density in a given spacetime region through the resonance occurring at a microscopic scale between two progressive waves of a^{ik} and one progressive wave of s^{ik}. From the way kept in achieving this result one expects that the production of net generalized forces through resonant processes in which more than three waves are involved can be demonstrated if one pushes to higher order in $[b]$ the approximation with which $\mathbf{g}^{ik}$ and $B_{[ik]}$ are calculated.

7. A PARTICULAR EXAMPLE OF RESONANT POWER ABSORPTION OR EMISSION

The resonance condition $k_m^A \pm k_m^B \pm k_m^C = 0$ is just the four- dimensional expression of the quantum mechanical rule known in the particular case of the frequencies as Bohr's condition. According to quantum physics this resonance condition plays a fundamental rôle for the exchanges of energy and momentum going on within matter; despite the utter differences in the variables involved and in the physical interpretation, the same condition is found necessary for the appearance of a nonvanishing $< \mathbf{T}^m_{k;m} >$, when $\mathbf{g}^{ik}$ is endowed with the wavy microscopic structure prescribed by (37). We still need to prove that the generalized forces associated to the three-waves resonances can be really nonvanishing; we shall do so through a particular example, freely sketched after the theoretical model that wave mechanics provides for the elementary processes of power absorption and emission in matter: an "atom" at rest is supposed to execute simultaneously two normal vibrations whose angular frequencies ω_1 and ω_2, in keeping with the relativistic description, are very large when compared to the angular frequency ω of a "light wave" that interacts with the atomic system. We lack at present field equations for $\mathbf{g}^{ik}$ that could describe this microscopic occurrence in a consistent way; we shall limit ourselves to render some of its tracts through the geometric objects of Einstein's unified field theory as follows.

With respect to the system of coordinates x^i, the metric s^{ik} is assumed to display very small deviations from the Minkowski form; its dependence on the coordinates is given by

$$s^{ik} = \eta^{ik} + u_A^{ik}(x^\mu) sin(\omega^A t + \varphi^A(ik)); \tag{51}$$

Greek letters again indicate the spatial coordinates, while $t \equiv x^4$ stands for the time coordinate, and $A = 1, 2$ labels the normal vibrations. The components of $u_A^{ik} = u_A^{ki}$ and their derivatives are assumed to vanish everywhere in spacetime, except within a world tube Π, whose spatial section Σ at x^4 =const. is compact and of atomic size; there the u_A^{ik} are so small that can be dealt with as first order infinitesimals. The positive angular frequencies ω_A are nearly equal, and we choose $\omega_1 \geq \omega_2$, while $\varphi^A(ik) = \varphi^A(ki)$ represent constant phases, that can take different values in different components of u_A^{ik}. We assume further that a^{ik} can be written as

$$a^{ik} = b^{ik} + c^{ik}, \tag{52}$$

where

$$b^{ik} = v_A^{ik}(x^\mu) sin(\omega^A t + \chi^A(ik)), \tag{53}$$

and

$$c^{ik} = d^{ik} sin(k_m x^m + \psi(ik)). \tag{54}$$

The components of $v_A^{ik} = -v_A^{ki}$ and their derivatives are everywhere vanishing, except within the world tube Π, where the v_A^{ik} can be treated as first order infinitesimals; $\chi^A(ik) = \chi^A(ki)$ are constant phases, that can be different for different components of v_A^{ik}; the normal vibrations of b^{ik} occur with the angular frequencies ω_1, ω_2 exhibited also by the metric s^{ik}. In (54), the components of $d^{ik} = -d^{ki}$ are small constants that can be considered as first order infinitesimals, while k_m is a four-vector, null with respect to the average metric $\bar{s}_{ik} = \eta_{ik}$, and $\psi(ik) = \psi(ki)$ are constant phases; therefore c^{ik} can behave as the components of **D** and **H** do, according to Maxwell's theory, for an electromagnetic plane wave in vacuo; furthermore, k_m is so chosen that the wavelength of c^{ik} is large with respect to the spatial extension of Σ and the positive angular frequency $k_4 \equiv \omega$ is very small with respect to both ω_1 and ω_2, its order of magnitude being the same as for the difference $\omega_1 - \omega_2$. We assume eventually that in our units $|s^{ik}{}_{,\mu}|$ and $|b^{ik}{}_{,\mu}|$ are small, when compared to $|s^{ik}{}_{,4}|$ and to $|b^{ik}{}_{,4}|$, since in the relativistic wavefunction of an atom the characteristic length for the spatial dependence is the Bohr radius, while the scale of the time dependence is provided by the Compton period.

The power absorption or emission by the "atom" will be detected through the average $< \mathbf{T}^m_{4;m} >$ extended to a box Ω which encloses the world tube Π for a span of the time coordinate that is very long with respect to the period $T = 2\pi/\omega$ of c^{ik} ; in these conditions the contribution to the average coming from the terms that are globally differentiated can be disregarded. Then we can avail of (41) and write

$$< \mathbf{T}^m_{4;m} >= -(1/16\pi) < [(-s)^{1/2}{}_{,4}(b^{im} + c^{im}) + (-s)^{1/2}(b^{im} + c^{im})_{,4}] B_{[im]} > . \tag{55}$$

This can hardly be called a macroscopic average, but one can readily imagine the extension of the present argument to an assembly of independent "atoms". A calculation adequate to reveal a resonant absorption or emission of power can be done through the approximation scheme of the previous Section; again, resonant processes in which only two oscillations take part are ruled out, and the next available possibility is a resonance in which three oscillations are involved, of which one and just one belongs to the metric.

Due to the choices done for the time dependence of s^{ik} and of a^{ik}, a resonant three-waves absorption or emission of power can only occur if $\omega = \omega_1 - \omega_2$, and the only terms in $< \mathbf{T}^m_{4;m} >$ of relevance in this process will be those that are written as a triple product, in which one factor is provided by s_{ik}, a second one by b^{ik}, and a third one by c^{ik}; of these terms, the ones that will contribute with the greatest strength will be those whose generic form reads

$$s_{ab,4,4}b^{cd}{}_{,4}c^{ef} \quad or \quad s_{ab,4}b^{cd}{}_{,4,4}c^{ef}, \tag{56}$$

i.e. those terms in which s_{ik} and b^{ik}, whose dependence on time was assumed to be faster than the spatial one, and also much faster than the spacetime dependence of c^{ik}, are collectively differentiated as many times as possible with respect to x^4.

The first and the last addendum at the right-hand side of (55) cannot produce terms with the forms (56) and can be dropped if one wishes to retain only the largest contributions, as we shall do. By availing of (A5) one finds the approximate expression

$$\begin{aligned} < \mathbf{T}^m_{4;m} > = &-(1/16\pi) < \eta^{pq}s_{pq,4}[(1/4)c^{\lambda\mu}b_{\lambda\mu,4,4} + (2/3)c^{4\mu}b_{4\mu,4,4}] \\ &+ b^{4\mu}{}_{,4}[c^4{}_\mu s_{44,4,4} + c^{4\rho}s_{\mu\rho,4,4} - (1/3)c^4{}_\mu\eta^{pq}s_{pq,4,4}] \\ &+ (1/2)b^{\lambda\mu}{}_{,4}[c^4{}_\mu s_{\lambda 4,4,4} - c^4{}_\lambda s_{\mu 4,4,4}] >, \end{aligned} \tag{57}$$

where indices have been lowered with η_{ik}. A specialization of this result that may be of some interest is attained if we assume that s^{ik}, whose form is given by (51), is conformally flat, i.e. if we can write also $s^{ik} = e^{-\sigma}\eta^{ik}$, with $|\sigma| \ll 1$. Then $< \mathbf{T}^m_{4;m} >$, after neglecting globally differentiated terms and by retaining only the largest contributions, gets the simple expression:

$$< \mathbf{T}^m_{4;m} >= -(1/16\pi) < \sigma_{,4}c^{ik}b_{ik,4,4} >, \tag{58}$$

from which it is recognized that a nonvanishing average absorption or emission of power can indeed take place, provided that the resonance condition $\omega = \omega_1 - \omega_2$ is satisfied.

The essential rôle played by the microscopic behaviour of the metric elicits a comment of a general character. If we consider the average generalized force produced by some fields defined on a rigid Minkowski background, we come up with an expression like $< T^m_{k,m} >$, where $T^m_{k,m}$ is the ordinary divergence of some energy tensor T^m_k; then, since $\int T^m_{k,m}d\Omega$ can be transformed into a surface integral over the boundary Δ of the domain of integration Ω, the detailed behaviour inside Ω, in particular a resonant behaviour of the fields that enter the definition of T^m_k, is completely irrelevant to the average: provided that the values of the fields and of their derivatives appearing in T^m_k were kept unaltered on Δ, the value of $< T^m_{k,m} >$ would remain the same also if the resonant behaviour of the fields were substituted with an incoherent one. The appearance of the covariant divergence $\mathbf{T}^m_{k;m}$ in the differential conservation laws of the general relativistic theories, as it occurs in (15), can make the difference: if $\mathbf{T}^m_{k;m}$ cannot be transformed into a sum of globally differentiated addenda, a direct relation becomes possible between a resonant microscopic behaviour of the fields, inclusive of the metric, and the average generalized force.

8. CONCLUDING REMARKS

Through particular examples obtained by imposing a priori some behaviour on the geometric objects of Einstein's unified field theory it has been shown how deep an influence a microscopic structure of the metric can exert on the macroscopic appearances that constitute the world of experience. Fluctuations of the metric with a very small amplitude and with a microscopic characteristic length are in fact capable of ruling the constitutive relation of macroscopic electromagnetism in nonconducting, nondispersive media; moreover, a microscopic wavy behaviour of the metric and of the field $\mathbf{a}^{ik}$ can result in the production of macroscopic generalized forces through three-waves resonance processes in which the wavevectors and the frequencies involved obey the very conditions that, according to quantum physics, rule the exchanges of energy and momentum occurring within matter.

It appears that the geometric objects of Einstein's unified field theory indeed offer entirely new opportunities for describing the macroscopic reality by starting from hypothetic microscopic structures and processes. The heuristic method adopted in the present paper is however of very limited scope: a priori assumptions for $\mathbf{g}^{ik}$ may suggest interesting possibilities, but in order to proceed further, field equations dictating the spacetime behaviour of $\mathbf{g}^{ik}$ need to be assigned and solved. As previously mentioned, such equations are presently lacking: unfortunately, we cannot rely on the ones proposed by Einstein [10] since, with the interpretation of the geometric objects proposed here, those equations imply that T_{ik}, $\mathbf{j}^i$ and K_{ikm} are vanishing everywhere. Only through field equations one can hope to develop a theory in which the outcomes of the previous Sections would be properly framed. Writing down sensible equations is of course a quite difficult task, but possibly not a desperate one: the whole wealth of experimental information of atomic and condensed matter physics is at our disposal as a guide in this endeavour.

APPENDIX

When terms up to $[kb^2]$ are retained, one writes

$$4\pi j_i = a_i{}^s{}_{;s} = s_{ip} a^{ps}{}_{,s} + (1/2)\eta_{ip}\eta^{sr} a^{pn} s_{sr,n}; \qquad (A1)$$

up to terms of magnitude $[k^2b^2]$, one finds

$$a_i{}^n S_{nk} = (1/2)\eta_{ip}\eta^{sr} a^{pn}(s_{nr,k,s} + s_{kr,n,s} - s_{nk,r,s} - s_{sr,n,k}), \qquad (A2)$$

$$a^{pq} S_{pqik} = a^{rn}(s_{ir,n,k} - s_{kr,n,i}), \qquad (A3)$$

and

$$\begin{aligned} a_{ik;a}^{\;;a} &= s_{ip} s_{kn} s^{aq} a^{pn}{}_{,a,q} \\ &+ \eta^{aq} a^{rn}{}_{,q}[\eta_{kn}(s_{ri,a} + s_{ai,r} - s_{ra,i}) - \eta_{in}(s_{rk,a} + s_{ak,r} - s_{ra,k})] \\ &- (1/2)\eta_{ip}\eta_{kn}\eta^{aq}\eta^{rs} a^{pn}{}_{,r}(2s_{as,q} - s_{aq,s}) \\ &+ (1/2)\eta^{aq} a^{rn}[\eta_{kn}(s_{ri,a} + s_{ai,r} - s_{ra,i}) - \eta_{in}(s_{rk,a} + s_{ak,r} - s_{ra,k})]_{,q}. \end{aligned} \qquad (A4)$$

Hence the sought for expression of $B_{[ik]}$ reads:

$$\begin{aligned} B_{[ik]} &= (1/6)[(s_{ip,k} - s_{kp,i}) a^{ps}{}_{,s} + s_{ip} a^{ps}{}_{,s,k} - s_{kp} a^{ps}{}_{,s,i}] \\ &+ (1/12)\eta^{sr}[\eta_{ip}(a^{pn} s_{sr,n})_{,k} - \eta_{kp}(a^{pn} s_{sr,n})_{,i}] \\ &+ (1/2)\big\{(1/2)\eta^{sr} a^{pn}[\eta_{ip}(2s_{kr,n,s} - s_{sr,n,k}) - \eta_{kp}(2s_{ir,n,s} - s_{sr,n,i})] \\ &+ a^{rn}(s_{ir,n,k} - s_{kr,n,i}) + s_{ip} s_{kn} s^{aq} a^{pn}{}_{,a,q} \\ &+ \eta^{aq} a^{rn}{}_{,q}[\eta_{kn}(s_{ri,a} + s_{ai,r} - s_{ra,i}) - \eta_{in}(s_{rk,a} + s_{ak,r} - s_{ra,k})] \\ &- (1/2)\eta_{ip}\eta_{kn}\eta^{aq}\eta^{rs} a^{pn}{}_{,r}(2s_{as,q} - s_{aq,s})\big\}. \end{aligned} \qquad (A5)$$

Two trigonometric relations are recalled for convenience:

$$\begin{aligned} cos\alpha sin\beta sin\gamma = -(1/4)[&cos(-\alpha+\beta+\gamma) - cos(\alpha-\beta+\gamma) \\ &- cos(\alpha+\beta-\gamma) + cos(\alpha+\beta+\gamma)], \end{aligned} \qquad (A6)$$

$$\begin{aligned} cos\alpha cos\beta cos\gamma = (1/4)[&cos(-\alpha+\beta+\gamma) + cos(\alpha-\beta+\gamma) \\ &+ cos(\alpha+\beta-\gamma) + cos(\alpha+\beta+\gamma)]. \end{aligned} \qquad (A7)$$

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