A FOUR-DIMENSIONAL HOOKE'S LAW CAN ENCOMPASS LINEAR ELASTICITY AND INERTIA

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Abstract. The question is examined, whether the formally straightforward extension of Hooke's time-honoured stress-strain relation to the four dimensions of special and of general relativity can make physical sense. The four-dimensional Hooke's law is found able to account for the inertia of matter; in the flat space, slow motion approximation the field equations for the "displacement" four-vector field ξ^i can encompass both linear elasticity and inertia. In this limit one just recovers the equations of motion of the classical theory of elasticity.

1. Introduction

After having fostered the birth of the four-dimensional approach to special relativity [1], macroscopic electromagnetism has found its natural expression in the four dimensional language of general relativity, both *in vacuo* [2] and in matter [3], [4]. Its field quantities and its field equations have achieved a canonical form, that can be summarized as follows [5]: the unconnected space-time manifold suffices for writing Maxwell's equation in the naturally invariant form:

$$\mathbf{H}^{ik}_{\ ,k} = \mathbf{s}^i, \tag{1.1}$$

$$F_{[ik,m]} = 0, \tag{1.2}$$

where $\mathbf{H}^{ik}$ is a skew, contravariant tensor density that represents the electric displacement and the magnetic field, while F_{ik} is a covariant skew tensor that accounts for the electric field and for the magnetic induction. Even if the four-current density $\mathbf{s}^i$ is given and the co-ordinate system is fixed, these equations are not sufficient for determining both $\mathbf{H}^{ik}$ and F_{ik}. They fulfil the two identities $\mathbf{H}^{ik}_{\ ,k,i} = 0$, which ensures the conservation of the electric four-current, and $\mathbf{e}^{ikmn}F_{[ik,m],n} = 0$, where $\mathbf{e}^{ikmn}$ is the totally antisymmetric tensor density of Ricci and Levi Civita. By simply counting the components of the fields, the equations and the identities one gathers that (1.1) and (1.2) need to be complemented by the so-called constitutive equation, *i. e.* by some tensor equation that uniquely defines for instance $\mathbf{H}^{ik}$ in terms of F_{ik} and of whatever fields may be needed in describing the features of the

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electromagnetic medium. For a linear medium the constitutive equations can be written as [3]:

$$\mathbf{H}^{ik} = \frac{1}{2}\mathbf{X}^{ikmn}F_{mn}; \tag{1.3}$$

the properties of the medium are specified by the four-index tensor density $\mathbf{X}^{ikmn}$. To complete this schematic picture of macroscopic electromagnetism, one may add that, once the constitutive equation is given, Maxwell's equations can be solved in terms of the four-vector potential φ_i. If F_{ik} is defined as:

$$F_{ik} \equiv \varphi_{k,i} - \varphi_{i,k}, \tag{1.4}$$

equation (1.2) is satisfied. Then the constitutive equation (1.3) takes the form:

$$\mathbf{H}^{ik} = \frac{1}{2}\mathbf{X}^{ikmn}(\varphi_{n,m} - \varphi_{m,n}) \tag{1.5}$$

and the four components of φ_i are determined, up to a gauge transformation, by the four equations (1.1). One may well wonder why the introduction to a paper announced to deal with elasticity and inertia begins with these electromagnetic reminiscences. But equation (1.5), this pervasive, general relativistic, four-dimensional constitutive equation is the antisymmetric counterpart of an equally pervasive, although pre-relativistic, three-dimensional and symmetric constitutive equation:

$$\Theta^{\lambda\mu} = \frac{1}{2}C^{\lambda\mu\rho\sigma}(\xi_{\rho,\sigma} + \xi_{\sigma,\rho}). \tag{1.6}$$

This is the law of linear elasticity, disclosed [6] by Hooke in 1678 with the words "*ut tensio sic vis*". It is given here by using the language of three-dimensional tensors and of tensor analysis, that were just invented to cope with the far-reaching developments stemmed from Hooke's discovery. In this way all the analogies between (1.5) and (1.6) are made apparent: the four-potential $\varphi_i(x^k)$ is the four-dimensional counterpart of the displacement vector $\xi_\rho(x^\lambda)$, the deformation tensor

$$S_{\rho\sigma} \equiv \frac{1}{2}(\xi_{\rho,\sigma} + \xi_{\sigma,\rho}) \tag{1.7}$$

is the symmetric, three-dimensional counterpart of F_{ik}, while the symmetric, three-dimensional stress tensor $\Theta^{\lambda\mu}$ of (1.6) is replaced in (1.5) with the skew, four-dimensional tensor density $\mathbf{H}^{ik}$. One might indulge in disclosing further analogies that betray the mechanistic origin of electromagnetism, but a question comes to the mind: have equation (1.6) and the classical theory of elasticity found a generally accepted reformulation in the framework of special and of general relativity, as it has occurred with electromagnetism? Even a cursory inspection of the literature shows that the answer is negative. This problem has attracted considerable attention in the years soon after 1905, as it is testified *e. g.* by the seminal paper of Herglotz on the mechanics of deformable bodies [7]. After 1915 Nordström resumed [3] Herglotz' approach and translated it in a general relativistic form, but, to our knowledge, the interest in the issue sinked, and only resurfaced at the time when Weber undertook [8] his studies on the detection of gravitational waves. In classical elasticity, the deformation of an elastic body is measured relative to a natural unstrained state. Synge found it difficult to carry over this idea into a pseudo-Riemannian space-time and decided [9] that the sentence "rate of change of stress linear function of rate of strain" should replace, in general relativity, Hooke's Latin dictum. Rayner resurrected the concept of reference state under the form of a reference metric [10] and postulated the following relativistic variant of Hooke's law:

$$\Sigma_{ij} \equiv \frac{1}{2} c_{ij}^{\ \ kl} (\gamma_{kl} - \gamma^0_{kl}) \tag{1.8}$$

where the auxiliary metric

$$\gamma^{ik} = g^{ik} + u^i u^k \tag{1.9}$$

defined in terms of the true metric g_{ik} and of the four-velocity u^i, and its reference counterpart γ^0_{ik} are availed of. Despite its four-dimensional clothing, equation (1.8) defines an effectively three-dimensional stress tensor: since both the auxiliary metric and its reference countepart are orthogonal to the four-velocity, one finds

$$\Sigma_{ij} u^j \equiv 0, \tag{1.10}$$

as it is appropriate for describing the elastic stress. Carter and Quintana extended Rayner's theory to a non-linear regime [11], as it is necessary when dealing with astrophysical situations. Nordström derived [3] his equations of motion from Hamilton's principle and showed their coincidence with the ones obtained through the general relativistic law of conservation for the energy tensor

$$\mathbf{T}^{ik}_{;k} = 0. \tag{1.11}$$

The later authors mentioned above and many others [12]-[17] instead relied only on equation (1.11) for setting up their equations of motion. There is however a notable exception in the Lagrangian formulation given [18] by Kijowski and Magli.

2. A really four-dimensional extension of Hooke's law

The diverse proposals for a general relativistic reformulation of Hooke's law all adopt an effectively three-dimensional stress-strain relation. It seems therefore natural to wonder whether a truly four-dimensional Hooke's law can have some physical sense. The general relativistic extension of equation (1.6) is formally immediate: in a pseudo-Riemannian space-time whose metric is g_{ik} one considers a contravariant four-vector field $\xi^i(x^k)$ that aims at representing some "displacement", and uses its covariant counterpart to define a four-dimensional "deformation" tensor

$$S_{ik} = \frac{1}{2}(\xi_{i;k} + \xi_{k;i}). \tag{2.1}$$

A four-dimensional "stiffness" tensor density $\mathbf{C}^{iklm}$ is then introduced; it will be symmetric in both the first pair and the second pair of indices, since it will be used for producing a "stress-momentum-energy" tensor density

$$\mathbf{T}^{ik} = \mathbf{C}^{iklm} S_{lm}, \tag{2.2}$$

through the four-dimensional extension of Hooke's law. If $\mathbf{T}^{ik}$ is meant to be the overall energy tensor density, it must obey the four-dimensional conservation law (1.11), that of course has to do with the equations of motion of the field $\xi^i(x^k)$. From a formal standpoint, this is nearly the end of the story. The real question is: can this abstract scheme find a physical interpretation, at least in some limit condition?

Let us start inquiring whether the vector field $\xi^i(x^k)$ can admit of the following physical meaning. One considers a co-ordinate system such that, at a given event:

$$g_{ik} = \eta_{ik} \equiv diag(1, 1, 1, -1), \tag{2.3}$$

while the Christoffel symbols are all vanishing, and the components of the four-velocity of matter are

$$u^1 = u^2 = u^3 = 0, \ u^4 = 1. \tag{2.4}$$

In this co-ordinate system we suppose it possible to measure, at the chosen event, both the components of the spatial displacement of the elastic medium from some relaxed condition and the proper time, that we interpret as temporal displacement. If these four quantities can be determined at any event with this procedure, they can be used to define, through the appropriate transformations, the four-vector field $\xi^i(x^k)$ in an arbitrary co-ordinate system. Let us assume that the field $\xi^i(x^k)$ that we have previously introduced in a purely formal way really allow for this physical interpretation. Then, if the material is unstrained in the ordinary sense, the only nonvanishing component of S_{ik} should be S_{44}, whose value should be -1. This remark suggests defining the four-velocity u^i of matter through the equation

$$\xi^i_{;k} u^k = u^i. \tag{2.5}$$

A necessary condition for the above definition to hold is:

$$det(\xi^i_{;k} - \delta^i_k) = 0. \tag{2.6}$$

This shall be one equation that the field ξ^i must satisfy; the number of independent components of ξ^i will thereby be reduced to three. We observe that the very definition of a four-velocity field $u^i(x^k)$ by starting from the field ξ^i is not ensured *a priori*; its possible existence will depend on the physical properties of the model that one considers.

A four-dimensional "stiffness" tensor C^{iklm} possibly endowed with physical meaning can be built as follows. We assume that in the locally Minkowskian rest frame defined above the only nonvanishing components of C^{iklm} are $C^{\lambda\nu\sigma\tau}$, with the tentative rôle of elastic moduli, and

$$C^{4444} = -\rho, \tag{2.7}$$

where ρ measures the rest density of matter (energy). We need to write the four-dimensional "stiffness" tensor in an arbitrary co-ordinate system. The task can be easily accomplished if matter is isotropic when looked at in the above mentioned Minkowskian rest frame. Just for the sake of simplicity, we shall deal henceforth only with this case. One avails of the auxiliary metric (1.9); then the part of C^{iklm} stemming from the ordinary elasticity of the isotropic medium can read [15]

$$C^{iklm}_{el.} = -\lambda\gamma^{ik}\gamma^{lm} - \mu(\gamma^{il}\gamma^{km} + \gamma^{im}\gamma^{kl}), \tag{2.8}$$

where λ and μ are assumed to be constants. The part of C^{iklm} accounting for the inertial term reads instead

$$C^{iklm}_{in.} = -\rho u^i u^k u^l u^m. \tag{2.9}$$

When no other fields are present, the tensor density $\mathbf{T}^{ik}$ defined by equation (2.2) must fulfil the four conditions (1.11). Since u^i is defined in terms of ξ^i and of the metric tensor g_{ik} via equation (2.5), imposing the four conditions (1.11) means postulating four nonlinear partial differential equations that, for a given g_{ik}, must be obeyed by the three independent components of the field ξ^i and by the scalar field ρ. Thanks to (2.8) and (2.5), the putative elastic part of the energy tensor T^{ik} for isotropic matter reads:

$$\begin{aligned} T^{ik}_{el.} = C^{iklm}_{el.} S_{lm} &= -\lambda(g^{ik} + u^i u^k)(\xi^m_{;m} - 1) \\ &-\mu[\xi^{i;k} + \xi^{k;i} + u_l(u^i\xi^{l;k} + u^k\xi^{l;i})], \end{aligned} \tag{2.10}$$

and is orthogonal to the four-velocity, as an intrinsically three-dimensional tensor should be, while the supposedly inertial part of T^{ik} turns out to be effectively so. In fact, due to equation (2.5):

$$T^{ik}_{in.} = C^{iklm}_{in.} S_{lm} = \rho u^i u^k. \tag{2.11}$$

When $\lambda = \mu = 0$ we have no elasticity at all, and the field equations (1.11) reduce to:

$$\{\rho u^i u^k\}_{;k} = \{\rho u^k\}_{;k} u^i + \rho u^i{}_{;k} u^k = 0. \tag{2.12}$$

By contracting the last equation with u_i one finds:

$$\{\rho u^k\}_{;k} = 0. \tag{2.13}$$

This equation is convenient for finding ρ, once the problem of motion, entirely written in kinematic terms:

$$u^i{}_{;k} u^k = 0, \tag{2.14}$$

has been solved. The world lines of matter are then timelike geodesics of the spacetime whose metric is g_{ik}. Once the geodesic equations are solved for u^i also the displacement field ξ^i is in principle determined through the equations (2.5) and (2.6). In the limit case when equation (2.3) holds everywhere the world lines of matter are straight and by performing a Lorentz transformation we can put matter everywhere to rest in the co-ordinate system, say, $x^i = (x, y, z, t)$. Then equation (2.4) is satisfied everywhere; we can fulfil both (2.5) and (2.6) by posing:

$$\xi^i = (0, 0, 0, t). \tag{2.15}$$

3. The classical theory of elasticity as limit case

Let us hold fixed both the co-ordinate system and the metric selected above, and assume that now the constants λ and μ are nonvanishing, but that u^i differs very slightly from its rest form (2.4). We deal with the components u^ρ as with first-order infinitesimal quantities, while of course the increment to u^4 shall be infinitesimal at second order. Since one expects that $\xi^\nu_{,\rho}$, like ξ^ν, and $\xi^4_{,\rho}$ will be at most first order infinitesimal quantities, the first order approximation to equation (2.5) requires

$$u^\nu = \xi^\nu_{,4}, \tag{3.1}$$

and

$$u^4 = \xi^4_{,4} = 1. \tag{3.2}$$

From the above remark about $\xi^i_{,\rho}$ and from equations (3.1), (3.2) one gathers that $\xi^i_{,k}$ will fulfil equation (2.6) to the required first order. Let us check the changes occurred to equation (2.13), that expressed the conservation of matter when ordinary elasticity was absent. One writes:

$$T^{ik}_{;k} u_i = (T^{ik} u_i)_{;k} - T^{ik} u_{i;k} = 0 \tag{3.3}$$

and, since

$$T^{ik}_{el.} u_k = 0, \tag{3.4}$$

equation (3.3) comes to read:

$$(-\rho u^k)_{;k} - T^{ik}_{el.} u_{i;k} = 0. \tag{3.5}$$

While the inertial term of this equation is a first-order infinitesimal, the putative elastic term contains only infinitesimal quantities of a higher order. Therefore equation (2.13) still holds, with the required approximation. We write now the field equations (1.11) by retaining only the first-order terms. The equations for $i = 1, 2, 3$ are:

$$\begin{aligned} &\{\rho u^\nu u^k - \lambda(\eta^{\nu k} + u^\nu u^k)(\xi^m_{,m} - 1) \\ &-\mu[\xi^{\nu,k} + \xi^{k,\nu} + u_l(u^\nu \xi^{l,k} + u^k \xi^{l,\nu})]\}_{,k} = 0. \end{aligned} \tag{3.6}$$

Due to (2.13) and (2.5), one can rewrite these equations as

$$\rho u^{\nu}{}_{,4} u^4 = \lambda \xi^{m,\nu}_{,m} + \mu(\xi^{\nu,\rho} + \xi^{\rho,\nu})_{,\rho}; \tag{3.7}$$

one eventually obtains, to the required order:

$$\rho \xi^{\nu}_{,4,4} = \lambda \xi^{m,\nu}_{,m} + \mu(\xi^{\nu,\rho} + \xi^{\rho,\nu})_{,\rho}. \tag{3.8}$$

Equation (3.2) says that $\xi^4_{,4}$ can differ from unity at most for second order infinitesimal terms; hence, with the required accuracy, $\xi^{\rho,\nu}_{,\rho}$ can be substituted for $\xi^{m,\nu}_{,m}$ in the previous equations, that come to read:

$$\rho \xi^{\nu}_{,4,4} = \lambda \xi^{\rho,\nu}_{,\rho} + \mu(\xi^{\nu,\rho} + \xi^{\rho,\nu})_{,\rho}. \tag{3.9}$$

The field equation for $i = 4$ reads:

$$\begin{aligned} &\{\rho u^4 u^k - \lambda(\eta^{4k} + u^4 u^k)(\xi^m_{,m} - 1) \\ &-\mu[\xi^{4,k} + \xi^{k,4} + u_l(u^4 \xi^{l,k} + u^k \xi^{l,4})]\}_{,k} = 0. \end{aligned} \tag{3.10}$$

The elastic terms provide only second-order contributions; therefore only the inertial term survives and, to the required order, the last field equation reads

$$\{\rho u^4 u^k\}_{,k} = 0, \tag{3.11}$$

as it occurs for the geodesic motion. The conservation equation (2.13) could have been adopted as fourth equation as well. The three equations (3.9) and the conservation law (2.13) exactly match the corresponding equations of the classical theory of elasticity.

4. Conclusion

As far as one can understand from the results of the previous Section, postulating a truly four-dimensional Hooke's law does not seems to be a false step, since it allows to merge both linear elasticity and inertia into a sort of extended, four-dimensional elasticity. Investigating the content of this approach when the metric is highly nonflat and the relative speed of different portions of matter is comparable with the velocity of light is a mathematically difficult and physically useless undertaking, since one must expect that the linear approximation of the classical theory of elasticity will fail to account for the behaviour of matter under such extreme conditions. Writing the equations of motion of elastic matter in presence of a weak, wavy deviation of g_{ik} from flatness [1] is instead both mathematically affordable and slightly more useful: in the complete lack of experimental evidence about the way elastic matter interacts with gravitational waves, it will at least allow for a comparison with the theoretical predictions achievable through the diverse approaches recalled in the Introduction.

[1] The very concept of weak gravitational wave is not without pitfalls. As wittily stressed by Eddington [19], one can easily manifacture perturbations of the flat metric that "propagate" with the *speed of thought.*

Confronted with the bareness of the experimental landscape, one can however seek consolation in the sense of intellectual fulfilment that looking at known things from a new perspective always provides. The expectation aroused by the analogy between the four-dimensional constitutive equation of electromagnetism (1.5) and the three-dimensional expression (1.6) of the stress-strain relation of classical elasticity was not void. Viewing inertia as a kind of elasticity is indeed possible, and this possibility is not constrained to the weak field, slow motion approximation. The reformulation of the inertial term of T^{ik} was possible thanks to equations (2.5) and (2.6), whose rôle was central in forcing the four-vector field ξ^i to account for both inertia and ordinary, classical elasticity.

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