ELECTRODYNAMIC FORCES IN ELASTIC MATTER

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Abstract. A macroscopic theory for the dynamics of elastic, isotropic matter in presence of electromagnetic fields is proposed here. We avail of Gordon's general relativistic derivation of Abraham's electromagnetic energy tensor as starting point. The necessary description of the elastic and of the inertial behaviour of matter is provided through a four-dimensional generalisation of Hooke's law, made possible by the introduction of a four-dimensional "displacement" vector. As intimated by Nordström, the physical origin of electrostriction and of magnetostriction is attributed to the change in the constitutive equation of electromagnetism caused by the deformation of matter. The part of the electromagnetic Lagrangian that depends on that deformation is given explicitly for the case of an isotropic medium and the resulting expression of the electrostrictive force is derived, thus showing how more realistic equations of motion for matter subjected to electromagnetic fields can be constructed.

1. Introduction

According to a widespread belief cultivated by present-day physicists, general relativity exerts its sovereign power in the heavens, where it supposedly rules tremendous astrophysical processes and awesome cosmological scenarios, but it has essentially nothing to say about more down to earth issues like the physics of ordinary matter, as it shows up in terrestrial laboratories. This way of thinking does not conform to the hopes expressed by Bernhard Riemann in his celebrated *Habilitationsschrift* [1]. While commenting upon the possible applications to the physical space of his new geometrical ideas, he wrote:

> Die Fragen über das Unmeßbargroße sind für die Naturerklärung müsige Fragen. Anders verhält es sich aber mit den Fragen über das Unmeßbarkleine [1].

The latter is presently supposed to be the exclusive hunting ground for quantum physics, whose workings occur at their best in the flat space of Newton. The classical field theories, in particular classical electromagnetism, are believed to have accomplished their midwife task a long time ago. Although

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[1]The questions about the infinitely great are for the interpretation of nature useless questions. But this is not the case with the questions about the infinitely small.

they are still revered, since they act as cornerstones for the applied sciences, and also provide the very foundation on which quantum mechanics and quantum field theory do stand, no fundamental insights are generally expected from their further frequentation. This mind habit cannot subsist without a strenuous act of faith in the final nature of the present-day reductionist programme: since for all practical purposes we have eventually attained the right microscopic laws, getting from there the right macroscopic physics should be just a matter of deduction by calculation (for the ever growing army of computer addicts, a sheer problem of computing power). Given time and endurance, we should be able to account for all the observed phenomena just by starting from our very simple microscopic laws!

It is not here the place for deciding how much this bold faith in the capabilities of today's reductionist approach be strenghtened by its undeniable successes, and how much it depend on having tackled just the sort of problems that are most suited to such a method. However, when confronted with the end results of many reductionist efforts, the obdurate classicist cannot help frowning in puzzlement. While he expects to meet with macroscopic laws derived from the underlying microscopic postulates by a pure exercise of logic, the everyday's practice confronts him with a much less palatable food. At best he is presented with rather particular examples usually worked out from the sacred principles through the surreptitious addition of a host of subsidiary assumptions. In the worst cases he is forced to contemplate and believe sequels of colourful plots and diagrams, generated by some computing device in some arcane way that he is simply impotent at producing again. In the intention of their proponents, both the "analytic" instances and their numerical surrogates should provide typical examples of some supposedly general behaviour, really stemming from the basic tenets of the theory, and in many a case this lucky occurrence may well have occurred, since "God watches over applied mathematicians" [2]. Nevertheless, the longing of the classicist for macroscopic laws of clear conceptual ancestry that do encompass in surveyable form a large class of phenomena remains sadly disattended. He is led to remind of the pre-quantum era, when both the reductionist approaches and the macroscopic ones were believed to be equally important tools for the advancement of physical knowledge [2], and to wonder whether relegating the macroscopic field approach to the "phenomenological" dustbin was really a wise move. Before being removed from center stage by more modern approaches, the macroscopic field theory lived long enough for developing, in the hands of great natural philosophers and mathematicians, theoretical tools of a very wide scope that, if still remembered and cultivated, would be recognized to be very useful today.

[2] It is remarkable how the otherwise daring Minkowski kept a cautious attitude with respect to Lorentz' atomistic theory of electricity both in his fundamental paper [3] of 1908 and also in his "Nachlass" notes [4], posthumously edited and published by Max Born.

2. Electrodynamic forces in material media

One of the clearest instances in which recourse to macroscopic field theory proves to be a quite helpful research tool occurs when one tries to describe the electromagnetic forces in material media. Since the time of Lorentz this has been a very challenging task; reductionist approaches starting from classical mechanics and from vacuum electrodynamics, for reasons clearly spelled out *e. g.* by Ott [5], end up in a disappointing gamut of possibilities also when the program of a rigorous special relativistic derivation is tenaciously adhered to [6], [7]. To our knowledge, a derivation of the macroscopic forces exerted by the electromagnetic field on a material medium performed by availing of quantum electrodynamics as the underlying microscopic theory, that should be *de rigueur* in the reductionist programme, has never been undertaken [3].

Happily enough, the theoretical advance in the methods for producing the stress energy momentum tensor of non gravitational fields occurred with the onset of general relativity theory [11], [12] have allowed W. Gordon to find, through a clever reduction to the vacuum case of the latter theory [13], a clear-cut argument for determining the electromagnetic forces in matter that is homogeneous and isotropic in its local rest frame. We shall recall Gordon's result in the next Section, since extending his outcome to the case of an elastic medium is just the scope of the present paper.

3. Gordon's *reductio ad vacuum* of the constitutive equation of electromagnetism

We adopt hereafter Gordon's conventions [13] and assume that the metric tensor g_{ik} can be locally brought to the diagonal form

$$g_{ik} = \eta_{ik} \equiv diag(1, 1, 1, -1) \tag{3.1}$$

at a given event through the appropriate transformation of coordinates. According to the established convention [14] let the electric displacement and the magnetic field be represented by the antisymmetric, contravariant tensor density $\mathbf{H}^{ik}$, while the electric field and the magnetic induction are accounted for by the skew, covariant tensor F_{ik}. With these geometrical objects we define the four-vectors:

$$F_i = F_{ik}u^k, \ H_i = H_{ik}u^k, \tag{3.2}$$

where u^i is the four-velocity of matter. In general relativity a linear electromagnetic medium can be told to be homogeneous and isotropic in its rest frame if its constitutive equations can be written as

$$\mu H^{ik} = F^{ik} + (\epsilon\mu - 1)(u^i F^k - u^k F^i), \tag{3.3}$$

[3]By availing once more of the midwife abilities of classical field theory, the converse has instead been attempted: some forms of "phenomenological" classical electrodynamics in matter has been subjected to some quantisation process [8], [9], [10], thereby producing diverse brands of "phenomenological" photons.

where the numbers ϵ and μ account for the dielectric constant and for the magnetic permeability of the medium [13]. This equation provides the constitutive relation in the standard form:

$$\mathbf{H}^{ik} = \frac{1}{2}\mathbf{X}^{ikmn}F_{mn}, \tag{3.4}$$

valid for linear media [15]. Gordon observed that equation (3.3) can be rewritten as

$$\mu H^{ik} = \left[g^{ir} - (\epsilon\mu - 1)u^i u^r\right]\left[g^{ks} - (\epsilon\mu - 1)u^k u^s\right]F_{rs}. \tag{3.5}$$

By introducing the "effective metric tensor"

$$\sigma^{ik} = g^{ik} - (\epsilon\mu - 1)u^i u^k, \tag{3.6}$$

the constitutive equation takes the form

$$\mu\mathbf{H}^{ik} = \sqrt{g}\sigma^{ir}\sigma^{ks}F_{rs}, \tag{3.7}$$

where $g \equiv -\det(g_{ik})$. The inverse of σ^{ik} is

$$\sigma_{ik} = g_{ik} + \left(1 - \frac{1}{\epsilon\mu}\right)u_i u_k, \tag{3.8}$$

and one easily finds [13] that

$$\sigma = \frac{g}{\epsilon\mu}, \tag{3.9}$$

where $\sigma \equiv -\det(\sigma_{ik})$. Therefore the constitutive equation can be eventually written as

$$\mathbf{H}^{ik} = \sqrt{\frac{\epsilon}{\mu}}\sqrt{\sigma}\sigma^{ir}\sigma^{ks}F_{rs}. \tag{3.10}$$

4. Gordon's derivation of Abraham's energy tensor

This result is the basis of Gordon's argument: since, apart from the constant factor $\sqrt{\epsilon/\mu}$, equation (3.10) is the constitutive equation for electromagnetism in a general relativistic vacuum whose metric be σ_{ik}, the Lagrangian density that we shall use for deriving the laws of the field is:

$$\mathbf{L} = \frac{1}{4}\sqrt{\frac{\epsilon}{\mu}}\sqrt{\sigma}F^{(i)(k)}F_{ik} - \mathbf{s}^i\varphi_i, \tag{4.1}$$

where $\mathbf{s}^i$ is the four-current density, while φ_i is the potential four-vector that defines the electric field and the magnetic induction:

$$F_{ik} \equiv \varphi_{k,i} - \varphi_{i,k}. \tag{4.2}$$

We have adopted the convention of enclosing within round brackets the indices that are either moved with σ_{ik} and σ^{ik}, or generated by performing the Hamiltonian derivative with respect to the mentioned tensors. The position (4.2) is equivalent to asking the satisfaction of the homogeneous set of Maxwell's equations

$$F_{[ik,m]} = 0, \tag{4.3}$$

while equating to zero the variation of the action integral $\int \mathbf{L} d\Omega$ with respect to φ_i entails the fulfilment of the inhomogeneous Maxwell's set

$$\mathbf{H}^{ik}_{\ ,k} = \mathbf{s}^i. \tag{4.4}$$

In our general relativistic framework, we can avail of the results found by Hilbert and Klein [11], [12] for determining the energy tensor of the electromagnetic field. If the metric tensor of our pseudo-Riemannian space-time were σ_{ik}, Hilbert's method would provide the electromagnetic energy tensor by executing the Hamiltonian derivative of the Lagrangian density $\mathbf{L}$ with respect to that metric:

$$\delta \mathbf{L} \equiv \frac{1}{2} \mathbf{T}_{(i)(k)} \delta \sigma^{ik}, \tag{4.5}$$

and we would get the mixed tensor density

$$\mathbf{T}_{(i)}^{\ (k)} = F_{ir} \mathbf{H}^{kr} - \frac{1}{4} \delta_i^{\ k} F_{rs} \mathbf{H}^{rs}, \tag{4.6}$$

which is just the general relativistic form of the energy tensor density proposed by Minkowski in his fundamental work [3]. But g_{ik}, not σ_{ik}, is the true metric that accounts for the structure of space-time and, through Einstein's equations, defines its overall energy tensor. Therefore the partial contribution to that energy tensor coming from the electromagnetic field must be obtained by calculating the Hamiltonian derivative of $\mathbf{L}$ with respect to g_{ik}. After some algebra [13] one easily gets the electromagnetic energy tensor:

$$T_i^{\ k} = F_{ir} H^{kr} - \frac{1}{4} \delta_i^{\ k} F_{rs} H^{rs} - (\epsilon\mu - 1) \Omega_i u^k, \tag{4.7}$$

where

$$\Omega^i \equiv -\left(T_k^{\ i} u^k + u^i T_{mn} u^m u^n\right) \tag{4.8}$$

is Minkowski's "Ruh-Strahl" [3]. Since $\Omega^i u_i \equiv 0$, substituting (4.7) into (4.8) yields:

$$\Omega^i = F_m H^{im} - F_m H^m u^i = u_k F_m \left(H^{ik} u^m + H^{km} u^i + H^{mi} u^k\right), \tag{4.9}$$

and one eventually recognizes that T_{ik} is the general relativistic extension of Abraham's tensor [16] for a medium that is homogeneous and isotropic when looked at in its local rest frame. The four-force density exerted by the electromagnetic field on the medium shall be given by (minus) the covariant divergence of that energy tensor density:

$$\mathbf{f}_i = -\mathbf{T}_{i\ ;k}^{\ k}, \tag{4.10}$$

where the semicolon stands for the covariant differentiation done by using the Christoffel symbols built with the metric g_{ik}. Abraham's energy tensor is an impressive theoretical outcome, that could hardly have been anticipated on the basis of heuristic arguments. Quite remarkably, the so called Abraham's force density, that stems from the four-divergence of that tensor, has found experimental confirmation in some delicate experiences performed by G. B. Walker et al. [17], [18]. Despite this, Abraham's rendering of the electrodynamic forces is not realistic enough, for it does not cope with the long known phenomena of electrostriction and of magnetostriction. We need to find its generalization, and we shall start from considering the case of linear elastic media, to which Hooke's law applies. This task would be made formally easier if one could avail of a relativistic reformulation of the linear theory of elasticity; the next Section will achieve this goal through a four-dimensional formulation of Hooke's law [19] that happens to be rather well suited to our scopes.

5. A four-dimensional formulation of Hooke's law

By availing of Cartesian coordinates and of the three-dimensional tensor formalism, that was just invented to cope with its far-reaching consequences, Hooke's law *"ut tensio sic vis"* [20] can be written as

$$\Theta^{\lambda\mu} = \frac{1}{2}C^{\lambda\mu\rho\sigma}(\xi_{\rho,\sigma} + \xi_{\sigma,\rho}), \tag{5.1}$$

where $\Theta^{\lambda\mu}$ is the three-dimensional tensor that defines the stresses arising in matter due to its displacement, given by the three-vector ξ^{ρ}, from a supposedly relaxed condition, and $C^{\lambda\mu\rho\sigma}$ is the constitutive tensor whose build depends on the material features and on the symmetry properties of the elastic medium. It seems natural to wonder whether this venerable formula can admit of not merely a redressing, but of a true generalization to the four-dimensions of the general relativistic spacetime. From a formal standpoint, the extension is obvious: one introduces a four-vector field ξ^{i}, that should represent a four-dimensional "displacement", and builds the "deformation" tensor

$$S_{ik} = \frac{1}{2}(\xi_{i;k} + \xi_{k;i}). \tag{5.2}$$

A four-dimensional "stiffness" tensor density $\mathbf{C}^{iklm}$ is then introduced; it will be symmetric in both the first pair and the second pair of indices, since it will be used for producing a "stress-momentum-energy" tensor density

$$\mathbf{T}^{ik} = \mathbf{C}^{iklm}S_{lm}, \tag{5.3}$$

through the four-dimensional generalization of equation (5.1). It has been found [19] that this generalization can be physically meaningful, since it allows one to encompass both inertia and elasticity in a sort of four-dimensional elasticity. Let us consider a coordinate system such that, at a given event, equation (3.1) holds, while the Christoffel symbols are vanishing and the components of the four-velocity of matter are:

$$u^1 = u^2 = u^3 = 0, \ u^4 = 1. \tag{5.4}$$

We imagine that in such a coordinate system we are able to measure, at the chosen event, the three components of the (supposedly small) spatial displacement of matter from its relaxed condition, and that we adopt these three numbers as the values taken by ξ^{ρ} in that coordinate system, while the reading of some clock ticking the proper time and travelling with the medium will provide the value of the "temporal displacement" ξ^4 in the same coordinate system. By applying this procedure to all the events of the manifold where matter is present and by reducing the collected data to a common, arbitrary coordinate system, we can define the vector field $\xi^i(x^k)$. From such a field we shall require that, when matter is not subjected to ordinary strain and is looked at in a local rest frame belonging to the ones defined above, it will exhibit a "deformation tensor" S_{ik} such that its only nonzero component will be $S_{44} = \xi_{4,4} = -1$. This requirement is met if we define the four-velocity of matter through the equation

$$\xi^i_{;k} u^k = u^i. \tag{5.5}$$

The latter definition holds provided that

$$det(\xi^i_{;k} - \delta^i_k) = 0, \tag{5.6}$$

and this shall be one equation that the field ξ^i must satisfy; in this way the number of independent components of ξ^i will be reduced to three [4]. A four-dimensional "stiffness" tensor C^{iklm} possibly endowed with physical meaning can be built as follows. We assume that in a locally Minkowskian rest frame the only nonvanishing components of C^{iklm} are: $C^{\lambda\nu\sigma\tau}$, with the meaning of ordinary elastic moduli, and

$$C^{4444} = -\rho, \tag{5.7}$$

where ρ measures the rest density of matter. But of course we need defining the four-dimensional "stiffness" tensor in an arbitrary co-ordinate system. The task can be easily accomplished if the unstrained matter is isotropic when looked at in a locally Minkowskian rest frame, and this is just the occurrence that we have already studied from the electromagnetic standpoint in Sections 3 and 4. Let us define the auxiliary metric

$$\gamma^{ik} = g^{ik} + u^i u^k; \tag{5.8}$$

then the part of C^{iklm} accounting for the ordinary elasticity of the isotropic medium can be written as [21]

$$C^{iklm}_{el.} = -\lambda \gamma^{ik} \gamma^{lm} - \mu(\gamma^{il}\gamma^{km} + \gamma^{im}\gamma^{kl}), \tag{5.9}$$

where λ and μ are assumed to be constants. The part of C^{iklm} that accounts for the inertia of matter shall read instead

$$C^{iklm}_{in.} = -\rho u^i u^k u^l u^m. \tag{5.10}$$

[4] The fulfilment of equation (5.5) is only a necessary, not a sufficient condition for the field ξ^i to take up the tentative meaning that was envisaged above. The physical interpretation of the field ξ^i can only be assessed *a posteriori* from the solutions of the field equations.

The elastic part $T^{ik}_{el.}$ of the energy tensor is orthogonal to the four-velocity, as it should be [22]; thanks to equation (5.5) it reduces to

$$T^{ik}_{el.} = C^{iklm}_{el.} S_{lm} = -\lambda(g^{ik} + u^i u^k)(\xi^m_{;m} - 1)$$
$$-\mu[\xi^{i;k} + \xi^{k;i} + u_l(u^i \xi^{l;k} + u^k \xi^{l;i})], \tag{5.11}$$

while, again thanks to equation (5.5), the inertial part of the energy tensor proves to be effectively so, since

$$T^{ik}_{in.} = C^{iklm}_{in.} S_{lm} = \rho u^i u^k. \tag{5.12}$$

The energy tensor defined by summing the contributions (5.11) and (5.12) encompasses both the inertial and the elastic energy tensor of an isotropic medium; when the macroscopic electromagnetic field is vanishing it should represent the overall energy tensor, whose covariant divergence must vanish according to Einstein's equations [11], [12]:

$$T^{ik}_{;k} = 0. \tag{5.13}$$

Imposing the latter condition allows one to write the equations of motion for isotropic matter subjected to elastic strain [22]. We show this outcome in the limiting case when the metric is everywhere flat and the four-velocity of matter is such that u^ρ can be dealt with as a first order infinitesimal quantity, while u^4 differs from unity at most for a second order infinitesimal quantity. Also the spatial components ξ^ρ of the displacement vector and their derivatives are supposed to be infinitesimal at first order. An easy calculation [19] then shows that equation (5.6) is satisfied to the required first order, and that equations (5.13) reduce to the three equations of motion:

$$\rho\xi^\nu_{,4,4} = \lambda\xi^{\rho,\nu}_{,\rho} + \mu(\xi^{\nu,\rho} + \xi^{\rho,\nu})_{,\rho}, \tag{5.14}$$

and to the conservation equation

$$\{\rho u^4 u^k\}_{,k} = 0, \tag{5.15}$$

i. e., to the required order, they come to coincide with the well known equations of the classical theory of elasticity for an isotropic medium.

6. Electrostriction and magnetostriction in isotropic matter

Having provided that portion of the equations of motion of matter that stems from the inertial and from the elastic part of the energy tensor, we can go back to the other side of our problem: finding to what changes the electrodynamic forces predicted in isotropic matter by Gordon's theory must be subjected in order to account for electrostriction and for magnetostriction. Driven by a suggestion found in the quoted paper by Nordström [15], we attribute the physical origin of the electrostrictive and of the magnetostrictive forces to the changes that the constitutive relation (3.4) undergoes when matter is strained in some way. If one desires to represent explicitly the effect of a small spatial deformation on the constitutive relation of electromagnetism, one can replace (3.4) with a new equation, written in terms of the new tensor density $\mathbf{Y}^{ikpqmn}$, that can be chosen to be antisymmetric with respect to the first pair and to the last pair of indices, symmetric with respect to the second pair. This tensor density allows one to rewrite the constitutive relation as follows:

$$\mathbf{H}^{ik} = \frac{1}{2}\mathbf{Y}^{ikpqmn} S_{pq} F_{mn}, \tag{6.1}$$

where F_{ik} is defined by (4.2) and S_{ik} is given by (5.2). For the intended application to isotropic matter it is convenient to split the equation written above in two terms, one concerning the unstrained medium, that has already been examined in Sections 3 and 4, and one dealing with the spatial deformation proper. Due to equation (5.5) one finds

$$\frac{1}{2}u^p u^q(\xi_{p;q} + \xi_{q;p}) = u_p u^q \xi^p_{;q} = u_p u^p = -1, \tag{6.2}$$

and the part (3.5) of the constitutive equation valid for the isotropic unstrained medium can be rewritten as:

(6.3)

$$\mathbf{H}^{ik}_{(u.)} = -\frac{\sqrt{g}}{\mu}\left[g^{im} - (\epsilon\mu - 1)u^i u^m\right]\left[g^{kn} - (\epsilon\mu - 1)u^k u^n\right] u^p u^q S_{pq} F_{mn}.$$

For producing the part of the constitutive equation that deals with the effects of the spatial deformation proper, we recall that an arbitrary deformation will bring the medium, which is now supposed to be isotropic when at rest and unstrained, into a generic anisotropic condition. When the magnetoelectric effect is disregarded [5], the electromagnetic properties of an anisotropic medium can be summarised, as shown *e. g.* by Schöpf [24], by assigning two symmetric four-tensors $\zeta_{ik} = \zeta_{ki}$ and $\kappa_{ik} = \kappa_{ki}$, whose fourth row and column vanish in a coordinate system in which matter happens to be locally at rest. This property finds tensorial expression in the equations

$$\zeta_{ik} u^k = 0, \ \ \kappa_{ik} u^k = 0; \tag{6.4}$$

ζ_{ik} has the rôle of dielectric tensor, while κ_{ik} acts as inverse magnetic permeability tensor. Let η^{iklm} be the Ricci-Levi Civita symbol in contravariant form, while η_{iklm} is its covariant counterpart. Then the generally covariant expression of the constitutive equation for the anisotropic medium reads [24]:

$$\mathbf{H}^{ik} = \sqrt{g}\left[(u^i \zeta^{km} - u^k \zeta^{im})u^n - \frac{1}{2}\eta^{ikrs} u_r \kappa_{sc} u_d \eta^{cdmn}\right] F_{mn}. \tag{6.5}$$

We shall avail of this equation to account for $\mathbf{H}^{ik}_{(s.)}$, i. e. for the part of $\mathbf{H}^{ik}$ produced, for a given F_{mn}, by the presence of ordinary strain in the otherwise isotropic medium. The tensors ζ^{ik} and κ^{ik} will now be given a new meaning: they represent henceforth only the changes in the dielectric properties and in the inverse magnetic permeability produced by the presence of strain. If the medium, as supposed, is thought to be isotropic when at rest and

[5]Such an effect indeed exists [23], but it is sufficiently *rara avis* to be neglected in the present context.

in the unstrained state, the dependence of ζ^{ik} and of κ^{ik} on S_{pq} will mimic the dependence on the four-dimensional deformation tensor exhibited by the elastic stress in an isotropic medium. One shall in fact write:

$$\zeta^{km} = \big[\alpha_1\gamma^{km}\gamma^{pq} + \alpha_2(\gamma^{kp}\gamma^{mq} + \gamma^{kq}\gamma^{mp})\big]S_{pq}, \tag{6.6}$$

where the constants α_1 and α_2 specify the electrostrictive behaviour of the isotropic medium. In the same way one is led to pose:

$$\kappa^{km} = \big[\beta_1\gamma^{km}\gamma^{pq} + \beta_2(\gamma^{kp}\gamma^{mq} + \gamma^{kq}\gamma^{mp})\big]S_{pq} \tag{6.7}$$

to account for the magnetostrictive behaviour; β_1 and β_2 are again the appropriate magnetostrictive constants for the isotropic medium. By availing of the definitions (6.6) and (6.7) one eventually writes

$$\mathbf{H}^{ik}_{(s.)} = \sqrt{g}\big[(u^i\zeta^{km} - u^k\zeta^{im})u^n - \frac{1}{2}\eta^{ikrs}u_r\kappa_{sc}u_d\eta^{cdmn}\big]F_{mn} \tag{6.8}$$

for the part of $\mathbf{H}^{ik}$ due to the ordinary strain. The overall $\mathbf{H}^{ik}$ is:

$$\mathbf{H}^{ik} = \mathbf{H}^{ik}_{(u.)} + \mathbf{H}^{ik}_{(s.)}, \tag{6.9}$$

and the two addenda at the right-hand side of this equation are the right-hand sides of equations (6.3) and (6.8); therefore the overall constitutive relation has just the form intimated by equation (6.1) for a general medium. As we did when electrostriction and magnetostriction were neglected, we assume again that the Lagrangian density $\mathbf{L}$ for the electromagnetic field in presence of the four-current density $\mathbf{s}^i$ shall read:

$$\mathbf{L} = \frac{1}{4}\mathbf{H}^{ik}F_{ik} - \mathbf{s}^i\varphi_i, \tag{6.10}$$

where φ_i is the four-vector potential, while $\mathbf{H}^{ik}$ has the new definition (6.9). Maxwell's equations (4.3) and (4.4) are then obtained in just the same way as it occurred with the Lagrangian density (4.1). Like $\mathbf{H}^{ik}$, also $\mathbf{L}$ can be split into an "unstrained" part $\mathbf{L}_{(u.)}$, given by equation (4.1), and a term stemming from strain, that will be called $\mathbf{L}_{(s.)}$. The Hamiltonian differentiation of $\mathbf{L}_{(u.)}$ with respect to the metric g_{ik} produces the general relativistic version of Abraham's energy tensor, as we know from Section 4. For clearness, we will rewrite it here as

$$\big(\mathbf{T}^{ik}\big)_{(u.)} = \sqrt{g}\big[F^i{}_r H^{kr}_{(u.)} - \frac{1}{4}g^{ik}F_{rs}H^{rs}_{(u.)} - (\epsilon\mu - 1)\Omega^i u^k\big], \tag{6.11}$$

where Ω^i now reads:

$$\Omega^i = u_k F_m\big(H^{ik}_{(u.)}u^m + H^{km}_{(u.)}u^i + H^{mi}_{(u.)}u^k\big). \tag{6.12}$$

Let us now deal with the explicit form of $\mathbf{L}_{(s.)}$. For simplicity we shall do so when only electrostriction is present, *i. e.* when $\kappa^{ik} = 0$. In this case one writes:

$$\mathbf{L}_{(s.)} = \frac{1}{4}\mathbf{H}^{ik}_{(s.)}F_{ik} = \frac{1}{4}\sqrt{g}\big[(u^i\zeta^{km} - u^k\zeta^{im})u^n\big]F_{mn}F_{ik}, \tag{6.13}$$

where ζ^{km} is defined by (6.6). Due to the antisymmetry of F_{ik}, $\mathbf{L}_{(s.)}$ can be rewritten as

$$\mathbf{L}_{(s.)}=\frac{1}{2}\sqrt{g}u^iu^n\zeta^{km}F_{mn}F_{ik}=\frac{1}{2}\sqrt{g}u^iu^n\left[\alpha_1\gamma^{km}\gamma^{pq}\right.$$
$$\left.+\alpha_2(\gamma^{kp}\gamma^{mq}+\gamma^{kq}\gamma^{mp})\right]S_{pq}F_{mn}F_{ik}. \tag{6.14}$$

Thanks to equation (5.5) one finds from (6.6):

$$\zeta^{km}=\alpha_1(g^{km}+u^ku^m)(\xi^s_{;s}-1)+\alpha_2\left[\xi^{k;m}+\xi^{m;k}+u_s(u^k\xi^{s;m}+u^m\xi^{s;k})\right], \tag{6.15}$$

hence:

$$\mathbf{L}_{(s.)}=-\frac{1}{2}\sqrt{g}\zeta^{km}F_kF_m=\frac{1}{2}\sqrt{g}u^iu^n\left\{\alpha_1(g^{km}+u^ku^m)(\xi^s_{;s}-1)\right.$$
$$\left.+2\alpha_2\left[\xi^{k;m}+u_su^k\xi^{s;m}\right]\right\}F_{ik}F_{mn}. \tag{6.16}$$

But of course the expression $u^iu^kF_{ik}$ identically vanishes; therefore the previous equation reduces to:

$$\mathbf{L}_{(s.)}=\frac{1}{2}\sqrt{g}u^au^n\left[\alpha_1g^{bm}(\xi^s_{;s}-1)+2\alpha_2g^{sm}\xi^b_{;s}\right]F_{ab}F_{mn}. \tag{6.17}$$

In our path towards the equations of motion of elastic matter subjected to electrodynamic forces we are now confronted with two options. We could attempt evaluating the Hamiltonian derivative of $\mathbf{L}_{(s.)}$ with respect to g_{ik}, then add the resulting tensor density to the overall energy tensor density $\mathbf{T}^{ik}$, of which we already know the inertial term from (5.12) , the elastic part from (5.11), and the "unstrained" electromagnetic component (6.11). The vanishing divergence of $\mathbf{T}^{ik}$ would then provide the equations of motion for the fields ξ^i and ρ, once the appropriate substitutions have been made, in keeping with the definition (5.5) of the four-velocity u^i. This program meets however with a certain difficulty: it requires assessing the metric content of $\xi^i_{;k}$ through extra assumptions of arbitrary character.

An alternative way is however offered. One can determine directly, without extra hypotheses, the contribution to the generalized force density $\mathbf{f}_{i(s.)}$ stemming from electrostriction through the Euler-Lagrange procedure:

$$\mathbf{f}_{i(s.)}=\frac{\partial\mathbf{L}_{(s.)}}{\partial\xi^i}-\frac{\partial}{\partial x^k}\left(\frac{\partial\mathbf{L}_{(s.)}}{\partial{\xi^i}_{,k}}\right). \tag{6.18}$$

If the metric g_{ik} is everywhere given by equation (3.1), and the velocity is so small that u^ρ can be dealt with as a first order infinitesimal quantity, while u^4 differs from unity only for second order terms, the Lagrangian density (6.17) comes to read:

$$L_{(s.)}=\frac{1}{2}\alpha_1F_{4\sigma}F^{4\sigma}\xi^\rho_{,\rho}-\alpha_2F_{4\rho}F_{4\sigma}\xi^{\rho,\sigma}, \tag{6.19}$$

and the nonzero components of its Hamiltonian derivative with respect to ξ^i are:

$$\mathbf{f}_{\rho(s.)} = \frac{\delta L_{(s.)}}{\delta \xi^\rho} = -\frac{\partial}{\partial x^\nu}\left(\frac{\partial L_{(s.)}}{\partial \xi^\rho{}_{,\nu}}\right)$$

$$= -\frac{1}{2}\alpha_1\big(F_{4\sigma}F^{4\sigma}\big)_{,\rho} - \alpha_2\big(F_{4\rho}F^{4\nu}\big)_{,\nu}. \tag{6.20}$$

In the case of fluid matter α_2 is vanishing, and the expression of the force density given by this equation agrees with the one predicted long ago by Helmholtz with arguments about the energy of an electrostatic system [25], and vindicated by experiments [26], [27] performed much later.

7. Conclusive remarks

The results of the previous Sections can be availed of in several ways. The full general relativistic treatment would require a simultaneous solution of Einstein's equations, of Maxwell's equations and of the equations $\mathbf{T}^{ik}_{\ ;k} = 0$ fulfilled by the overall energy tensor, thereby determining in a consistent way the metric g_{ik}, the four-potential φ_i, the "displacement" four-vector ξ^i, and ρ. This approach is presently *extra vires*, due to our ignorance of the part of $\mathbf{T}^{ik}$ stemming from $\mathbf{L}_{(s.)}$, for the reason mentioned in the previous Section. Achievements of lesser consistency are instead at hand, like solving the equations for the electromagnetic field and for the material field described by ξ^i and ρ with a given background metric, or finding the motion of elastic matter with a fixed metric, while the electromagnetic field is evaluated as if electrostriction were absent. Obviously enough, the calculations become trivial when the metric is everywhere given by (3.1), while the motion of matter occurs with non relativistic speed. In the present paper we have required that matter be isotropic when at rest and unstrained, but this limitation was just chosen for providing a simple example. The theory can be extended without effort to cristalline matter exhibiting different symmetry properties, for which reliable experimental data have started accumulating in recent years.

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