ON THE RELATIVISTIC CONCEPT OF *BILDRAUM*

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Abstract. Levi-Civita's treatment of the relations between two different metrics referred to the same coordinate system allows to formulate a general definition of *Bildraum*. Weyl's representation with "canonical" cylindrical coordinates of the Schwarzschild manifold of a mass point gives a very clear illustration of the concept.

1. – The concept of *Bildraum* [1] (picture space, representative space, auxiliary space, *etc.* – the German term is due to Weyl [1*a*)]) is commonly employed in general relativity (GR), implicitly or explicitly. Basing on a particular instance, Synge [2] gave a detailed illustration of it. It seems however that a *general* definition of this important notion has not been formulated. We intend to give it.

2. – In Chapt. *VIII* of the book quoted in [1*b*)], Levi-Civita developed the mathematical theory of the relations between two *different* metrics referred to the *same* coordinate system. He remarked that "There is clearly no reason against assigning *in turn* [the italics are of the present authors – [3]] to the same analytical manifold two distinct metrical determinations, defined by the two quadratic forms [...]:

$$\text{(I)} \qquad \mathrm{d}s^2 = \sum_1^n {}_{ik}\, a_{ik}\, \mathrm{d}x_i\, \mathrm{d}x_k$$

$$\text{(II)} \qquad \mathrm{d}s'^2 = \sum_1^n {}_{ik}\, a'_{ik}\, \mathrm{d}x_i\, \mathrm{d}x_k \quad ."$$

And in a footnote he added: "As a geometrical interpretation, we can think of two distinct V_n's whose points are in one-to-one correspondence, so that a set of n values assigned to $x_1, x_2, \ldots, x_n$ can be represented either by a point P of one, or by the corresponding point P' of the other. E.g. a map and the surface of the earth are two V_n's with different metrics (one is Euclidean, the other not), and to every pair of values, φ (for the latitude), λ (for the longitude), correspond one point on the map and one point on the earth."

The very detailed treatment by Levi-Civita is not necessary for our aim. Therefore we limit ourselves to the interesting remark that the difference

${\Gamma^{l}}_{rs} - {\Gamma'^{l}}_{rs}$ between the Christoffel symbols of the above metrics (I), (II) is a *tensor* ${\varrho^{l}}_{rs}$.

For a general definition of *Bildraum* these considerations are sufficient. Of course, they must be applied to the manifolds of GR; in every problem we have the physical metric and the metric – generally a Euclidean one (three-dimensional, or four-dimensional) – of a suitable *Bildraum*. An important property of a *Bildraum* is the *faithful* correspondence with its Einsteinian manifold. In this way, the conclusions reached by a reasoning on the *Bildraum* can be immediately transferred to the Einsteinian manifold.

3. – Weyl's example (see Part B of [1*a*)]) of the cylindrical representation of the $\mathrm{d}s^2$ of the gravitational field created by a point-mass m (at rest) is enlightening.

For a right interpretation of Weyl's results, we remark that the first (1917, [1*a*)]) Weyl's conception of the singular surface $r = 2m$ of the standard form of solution to Schwarzschild problem, according to which the mass would be distributed on this surface, has revealed itself as untenable. And an analogous criticism holds for the corresponding singular surfaces of the isotropic and cylindrical coordinates. However, Weyl's idea is perfectly (and trivially!) valid for the *original* Schwarzschild's form of solution ($r_{\text{standard}} \rightarrow [r^3 + (2m)^3]^{1/3}$) and for Brillouin's form ($r_{\text{standard}} \rightarrow r + 2m$).

4. – To obtain the cylindrical form of solution to Schwarzschild problem, Weyl [1*a*)] proceeds as follows.

He starts from the standard (Hilbert-Droste-Weyl) form of the $\mathrm{d}s^2$ ($G = c = 1$):

$$\text{(1)}\quad \mathrm{d}s^2 = \left(1 - \frac{2m}{r}\right)\mathrm{d}t^2 - \left(1 - \frac{2m}{r}\right)^{-1}\mathrm{d}r^2 - r^2\left(\mathrm{d}\vartheta^2 + \sin^2\vartheta\,\mathrm{d}\varphi^2\right) \quad ;$$

with the substitution:

$$\text{(2)}\qquad r \rightarrow \left(1 + \frac{m}{2r}\right)^2 \quad ; \quad \Rightarrow \mathrm{d}r \rightarrow \left(1 + \frac{m^2}{4r^2}\right)\mathrm{d}r \quad ,$$

from which:

$$\text{(3)}\qquad \{r = 2m\} \rightarrow \left\{r = \frac{m}{2}\right\} \quad ,$$

he gets the metric in *isotropic* coordinates:

$$\text{(4)}\quad \mathrm{d}s^2 = \left[\frac{r - (m/2)}{r + (m/2)}\right]^2 \mathrm{d}t^2 - \left[1 + \frac{m}{2r}\right]^4 \cdot \left[\mathrm{d}r^2 + r^2(\mathrm{d}\vartheta^2 + \sin^2\vartheta\,\mathrm{d}\varphi^2)\right] \quad .$$

Observe that the surface area $4\pi r^2$ of the standard form (1) corresponds to the surface area $4\pi r^2[1 + (m/2r)]^4$ of metric (4).

The two space regions $r > (m/2)$ – external region – and $r < (m/2)$ – internal region – are on an equal footing from the mathematical standpoint. But from the physical standpoint, as Weyl affirms, things stand otherwise: "Bei analytischer Fortsetzung wird $[r - (m/2)]/[r + (m/2)]$ im Innern negativ, so daß also dort für einen ruhenden Punkt kosmische Zeit ($t = x_4$) und Eingenzeit gegenläufig sind. (in der Natur kann selbstverständlich nur immer ein bis an die singuläre Kugel $[r = m/2]$ }nicht heranreichendes Stück der Lösung verwirklicht sein."

Weyl makes now a transition to cylindrical coordinates ϱ, φ, z:

$$(5) \quad x = \varrho \cos\varphi\,; \quad y = \varrho \sin\varphi\,; \quad z = z\,; \quad (0 \le \varrho < \infty\,; \quad 0 \le \varphi < 2\pi)\,.$$

Clearly, this presupposes that *we have chosen a Euclidean three-dimensional* ***Bildraum***, so that the relations between the coordinates r, ϑ, φ of eq. (4) and the Cartesian orthogonal coordinates x, y, z are the customary ones. Eqs. (5) are referred to this ***Bildraum***. Then, with the following *conformal* transformation in the meridian half-plane of coordinates (5):

$$(6) \qquad \varrho + iz - \frac{(m/2)^2}{\varrho + iz} = \varrho_* + iz_* \quad ; \varphi = \varphi_* \quad ,$$

Weyl arrives at an interval $\mathrm{d}s^2$ written in the "canonical" cylindrical coordinates ϱ_*, z_*, φ:

$$(7) \qquad \mathrm{d}s^2 = f\,\mathrm{d}t^2 - h(\mathrm{d}z_*^2 + \mathrm{d}\varrho_*^2) - f^{-1}\varrho_*^2\,\mathrm{d}\varphi^2 \quad ,$$

where:

$$(8) \qquad f \equiv \frac{r_1 + r_2 - 2m}{r_1 + r_2 + 2m} \quad ; \quad h \equiv \frac{\{[(r_1 + r_2)/2] + m\}^2}{r_1 r_2} \quad ,$$

$$(9) \qquad r_1^2 \equiv \varrho_*^2 + (z_* - m)^2 \quad ; \quad r_2^2 \equiv \varrho_*^2 + (z_* + m)^2 \quad .$$

The singular surface $r = m/2$ of the isotropic metric (4), and the singular surface $r = 2m$ of the standard metric (1), correspond to the following segment (*Strecke*) of the z_*-axis:

$$(9) \qquad \varrho_* = 0 \quad ; \quad -m \le z_* \le +m \quad .$$

We reproduce here Weyl's Fig.2, p.140 [1*a*)].

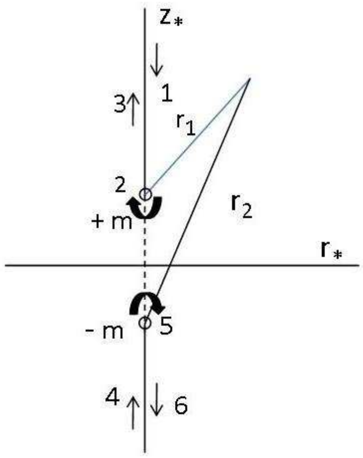

FIGURE 1. Weyl's Fig.2, [1*a*)]

It clarifies that there exists a perfect correspondence between the sphero-symmetrical form (4) and the "canonical" cylindrical form (7). The z_*-axis is cut along the half-lines $z_* > +m$ and $z_* < -m$; the cut (which is closed at infinite) must be followed according to the numbers and the arrows of the figure. The right half of the full meridian plane corresponds to the external part ($r > m/2$) of the isotropic metric, the left half to the internal one ($r < m/2$).

5. – We think that the knowledge of *all* the physically important forms of solution to Schwarzschild problem – *in particular*, of the original Schwarzschildian form (1916), of Brillouin's form (1923), and of Weyl's cylindrical form (1917) – would have preserved many physicists from believing in fictive interpretations of the surface $r = 2m$ of the standard form.

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REFERENCES

[1] : *a*) H. Weyl, *Ann. Physik*, **54** (1917) 117; *b*) T. Levi-Civita, *The Absolute Differential Calculus (Calculus of Tensors)* – (Dover Pubblications, Inc., Mineola, N.Y.) 2005, p. 408 *sqq.* (Originally published: London: Blackie and Son, 1926). The Italian edition: *Lezioni di calcolo differenziale assoluto* (A. Stock, Roma) 1925 does not contain Part III (theory of relativity).

[2] J.L. Synge, *Proc. Roy. Ir. Acad.*, **53A** (1950) 83.

[3] The words “in turn” represent the English translation of the Italian adverb “successivamente”: we see that Levi-Civita rejected (*ante litteram*!) the so-called *bimetric theory*, against which Einstein pronounced an “*anathema sit*” – cf., *e.g.*, footnote[2], p.76, of his “Autobiographisches” in *Albert Einstein*: *Philosopher-Scientist.* edited by P.A. Schilpp (Tudor Publishing Company, New York) 1949.
For a very clear formulation of the bimetric theory, see M. Kohler, *Z. Phys*, **139** (1952) 571; **134** (1953) 286 and 306. And M. v. Laue, *Die Relativitätstheorie*, Zweiter Band, 4. neubearbeitete Auflage (Friedr. Vieweg und Sohn, Braunschweig) 1956, sects. **54** and **55**. –

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