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Teaching Electromagnetism in elementary physics or upper high school courses

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Abstract. Traditionally, Electromagnetism is taught following the chronological development of the matter. The final product of this path is a presentation of Electromagnetism realized by adding one layer over another with the risk of transferring concepts and formulae from Electrostatics to Electrodynamics. In this paper, we suggest a new approach based on the idea that the matter should be presented within the conceptual framework of Maxwell-Lorentz-Einstein Electromagnetism. This approach is founded on the concept of a field as a primary theoretical entity and on the statement that a point charge produces, in general, an electric and a magnetic field and that the force exerted by these fields on a point charge is the Lorentz force. Developing this idea, one finds that macroscopic laws corroborated by experiments have a microscopic origin. It also follows that the electromotive force induced in a closed conducting circuit must be defined as the line integral of the Lorentz force on a unit positive charge. This definition leads to a local law of electromagnetic induction, Lorentz's invariant for rigid and filiform circuits. This law contrasts with what Feynman labeled as the "flux rule" —generally taught in textbooks and teaching practices— downgrading it from the status of physical law. Particular attention is given to the teaching dilemma of Maxwell's equations: ignore them, write them in integral form, or speak of them, focusing on their conceptual and physical meaning.

1. Introduction

Traditionally, in elementary physics and upper high school courses, Electromagnetism is taught following the chronological development of the matter (1). With the term "elementary physics courses", we refer to college courses whose level is intermediate between high school and university. As for high schools, the students' mathematical and physical background knowledge varies widely from country to country. As for Italy, we refer to the last three years of scientific high school.

(1) This approach is typical for first-level university textbooks.

The first topic is Electrostatics, based on Coulomb's law. Usually, the definition of the electric field derives from the concept of force. There are two typical approaches. One states the Coulomb law as

$$F = q\left(\frac{kQ}{r^2}\right), \tag{1}$$

where k is a constant depending on the unit system employed, Q is a large charge, and q is a small enough charge called test charge. The term in brackets depends only on the charge Q. Then, the norm of the electric field vector is defined as F/q, and $\vec{E} = \vec{F}/q$ gives the electric field vector. This approach is found, for instance, in [1], p. 46.

Otherwise, one can define the electric field "as the force exerted on a unit positive charge by a charged body". See, for instance [2], p. 421. If we step forward in time by about fifty years, we see that nothing has changed (see [3], pp. 630–635, [4], pp. 450-451). In these definitions the field concept is derived from that of force: it is not a primary concept.

The further development of the matter deals with continuous currents. The microscopic nature of currents is somewhat remembered; however, it generally does not enter into any calculation. Here, we have a conceptual discontinuity. While Coulomb's law talks about of point charges, their constitutive role in currents is overlooked.

The magnetic field is introduced by considering magnets or by recalling Ørsted's discovery of the deviation of a magnetized needle by a current-carrying wire. The experiments by Ampère allow us to discuss the magnetic forces exerted by current-carrying wires within a macroscopic description. The role of moving point charges as ultimate sources of the magnetic field needs to be recovered.

The magnetic component of Lorentz force is introduced as an experimental finding. Then, the force exerted by an electromagnetic field on a point charge becomes

$$\vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right), \tag{2}$$

where $\vec{v}$ is the charge velocity. Nonetheless, the induced electromotive force is defined as

$$\mathcal{E} = \oint \vec{E} \cdot \mathrm{d}\vec{l}, \tag{3}$$

instead of

$$\mathcal{E} = \oint \left(\vec{E} + \vec{v}_c \times \vec{B}\right) \cdot \mathrm{d}\vec{l}, \tag{4}$$

as the expression of Lorentz force implies and as argued in [5, 6] ($\vec{v}_c$ is the charge velocity).

Critical issues are constituted by electromagnetic induction and Maxwell's equations. For electromagnetic induction, textbooks, and teaching practices rely on the "flux rule". Feynman (see [7], pp. 17.1–17.3) and, more recently, Giuliani [6] have shown that the "flux rule" is only a calculation shortcut and not a physical law. As for Maxwell's equations, their differential form requires mathematical skills that are out of reach. Then, how to manage these two fundamental issues?

An epistemological stand accompanies this traditional approach, according to which experiments must induce physical laws. Moreover, the diffuse habit of referring to students' daily life and sensorial experiences obscures the role of the theories and the need for abstraction. The final product of this traditional path is a presentation of Electromagnetism realized by adding one layer over another with the risk of transferring concepts and formulae from Electrostatics to Electrodynamics. The definition of the induced electromotive force given by eq. (3) instead of the correct one (4) is a striking example.

The broad literature in Physics Education generally deals with students' difficulty understanding fundamental concepts or focuses on typical students' misconceptions or misunderstandings. These studies often rely on multiple-choice tests, sometimes integrated with interviews. The validity of multiple-choice tests as a means to evaluate students' understanding have been studied by many authors. See, for instance, [9] and the references therein. However, their utility in unrevealing students' misconceptions or misunderstandings seems out of doubt.

We shall discuss two papers particularly suited for our discourse. Sağlam and Millar used a multiple choice test administered to English and Turkish upper high school students [10]. They also interviewed a sample of Turkish students to determine the reasoning followed in answering the test. The questions concerned three fundamental topics of Electromagnetism: "magnetic field (caused by moving charges), magnetic forces (on moving charges and current-carrying wires), and electromagnetic induction" (see [10], p. 546). Sağlam and Millar found four types of difficulty in understanding these electromagnetic topics (see [10], p. 558):

1) inappropriate analogies between the effect of magnetic and electric field on electric charges,
2) an over-literal flow interpretation of magnetic field lines,
3) incorrect use of direct cause-effect reasoning in situations where it does not apply,
4) confusion between change, and rate of change, of variables (such as magnetic flux).

In their conclusions, Sağlam and Millar wrote (see [10], p. 564; our italics):

> By using samples from two countries, the study also shows a striking level of agreement in the questions (and hence perhaps the ideas) that students found most straightforward and most difficult. *This increases confidence that learning difficulties are due to inherent characteristics of the material, rather than stemming from the way it is taught (which is quite different in the two countries).*

Indeed, Maxwell's Electromagnetism, in its modern version —owing to the contributions by Lorentz and Einstein (Maxwell-Lorentz-Einstein Electromagnetism— MLE), is conceptually challenging for two fundamental reasons: it requires a theoretical approach centered on the concept of the field; it incorporates the special relativity result of a limited speed for material particles and physical interactions. These conceptual features are at stake with the Newtonian view, where forces-at-a-distance are the main actors, and every speed is allowed. Coulomb's law operates in a strictly Newtonian view. We underline that students' difficulties are enhanced by teaching practices — fueled by textbooks and syllabuses— based on the chronological development of the matter. A possible way out is teaching Electromagnetism within the MLE conceptual framework.

A relatively recent study by Zuza *et al.* [11] reinforces this working hypothesis. The authors used a test made of six conceptual free-response questions proposed to first- or second-year university students in three different European countries (Spain, Belgium and Ireland). Also in this study, the students' difficulties are independent of their country, "regardless of differences in their educational system and cultural background". The fact that people involved in the test were first years university students is not significant unless we assume that the misconceptions or misinterpretations surfaced were due to university teaching and that this teaching has completely canceled previous misconceptions or misinterpretations. After a careful discussion of the answers, the authors wrote:

> In conclusion, we believe that more attention should be paid to the specific characteristics of field theory and the difference between fields and forces, with particular emphasis on the conceptual interpretation of the interaction process rather than rules. Such an approach would guide students in the transition from a Newtonian to a Maxwellian viewpoint, underpinned by a changing view of the field from a calculational convenience to a physical entity.

The difficulty of substituting the Newtonian force-centered viewpoint with the field conceptual framework demands a change also in how we teach electromagnetic phenomena in high school or in elementary physics courses.

The proposal discussed in this paper requires abandoning the chronological development of the matter and presenting Electromagnetic phenomena to students within the conceptual framework of MLE.

This paper is organized as follows. Section 2 presents the main traits of our proposal. Section 2.1 deals with the concept of the field as a primary theoretical entity. Section 2.2 suggests how to introduce the idea that a point charge produces, in general, an electric and a magnetic field. Section 2.3 deals with the opportunity of introducing the vector potential in an elementary way. Section 2.4 treats the problem of electromagnetic induction. Section 2.5 discusses to what extent, if any, teachers should speak about Maxwell's equations. The reader will find a general discussion and conclusive remarks in the last sect. 3.

2. The broad lines of the proposal

The present proposal is not —and could not be— a receipt ready to use. It only highlights the main and interrelated conceptual features of MLE that could be transferred into high school or elementary physics teaching. It can be read at two levels:

as an occasion for refreshing the teachers' cultural background or as a guide for trying some changes in the teaching practices. We know the many constraints that limit the teachers creativity (at least in Italy): the syllabuses prepared by the Ministry of Education and the local tendency to standardize teaching practices in all the classrooms, with the adoption of the same textbooks, also in the view of preparing the students for the final state exam. Also, the textbook publishers' policy of supplying many developed didactic tools (lessons included) does not stimulate teachers' creativity. The proposal contemplates the possibility of using some formalism more complicated than the one commonly used. However, at each step of this kind, it is stressed that the important thing is the concept, not the formula accompanying it. The choice of formalism is left to the teacher, who must consider his teaching context ([2]).

An introductory discourse should take up again the difference between Galileo's and Einstein's relativity principles. Both principles require that physical laws have the same form in every inertial frame. However, while the former obeys Galileo's coordinates transformations, the latter obeys Lorentz's. The former allows physical interactions with infinite speed. Instead, the latter implies that physical interactions can propagate only with a finite velocity whose upper limit is light speed in a vacuum. The physical differences between the two views are well illustrated by considering the gravitational field produced by a mass in Newtonian mechanics and the electric field produced by a point charge in Electromagnetism. In Newton's gravitational theory, the gravitational field produced at the point $\vec{r}_1$ at the time t by a mass depends on the position $\vec{r}_2$ of the mass at the same time t (physical interactions propagate at infinite speed). Instead, in Electromagnetism, in a chosen inertial reference frame, the electric field produced at the point $\vec{r}_1$ at the time t by a moving point charge depends on the position of the charge at an earlier time, named "retarded time": to this retarded time also pertain a "retarded position", a "retarded velocity", and a "retarded acceleration" of the point charge.

Independently of the fact that teachers use Maxwell's equations in some form or not, they should emphasize that these equations have allowed, through the work of Hertz, Lorentz, and Einstein, the possibility of describing all electromagnetic phenomena in a vacuum in an axiomatic way. This step is, among others, necessary for correcting students' conception of physics, and in general of science, as a discipline founded essentially (if not only) on experiment. Consequently, teachers should talk about physicists' two principal methods to establish their discipline's laws: the inductive and the axiomatic method. This discourse should conveniently refer to the historical development of Electromagnetism. The inductive method was dominant during the nineteenth century. Referring to Faraday's fundamental contributions to electromagnetic induction, Maxwell wrote: "The method which Faraday employed in his researches consisted in a constant appeal to experiment as a mean of testing the

([2]) The physical and mathematical background knowledge of the students potentially involved in the experimentation varies widely from country to country. The teachers must decide what the extent to adapt the present proposal to their teaching conditions.

truth of his ideas, and a constant cultivation of ideas under the direct influence of experiment" (see [8], p. 163). Maxwell wrote his equations after having derived many laws from experimental results. Hertz, referring to Maxwell's equations, vindicated the importance of the axiomatic method with these words:

> These statements [Maxwell's equations] form, as far as the ether is concerned, the essential parts of Maxwell's theory. Maxwell arrived at them by starting with the idea of action-at-a-distance and attributing to the ether the properties of a highly polarisable dielectric medium. We can also arrive at them in other ways. *But in no way can a direct proof of these equations be deduced from experience. It appears most logical, therefore, to regard them independently of the way in which they have been arrived at, to consider them as hypothetical assumptions, and to let their probability depend upon the very large number of natural laws which they embrace* (see [12], *p. 138, italics added*).

Teachers should also draw students' attention to their study of Newtonian mechanics and thermodynamics within an axiomatic approach.

The present proposal rests on three cornerstones:

1) The use of the field concept as a primary theoretical entity and the necessity of introducing the field concept beginning with the gravitational interaction (In Italian scientific high schools, during the third year).
2) The idea that electromagnetic phenomena must be treated within the conceptual domain of Maxwell-Lorentz-Einstein Electromagnetism.
3) The statement that only local equations can be *interpreted* causally. This statement stems from special relativity, and it means that an equation is local if it connects two physical quantities at a given point at the same time t, or the equation connects a physical quantity at point $\vec{r}_1$ at the time t_1 to another physical quantity at the point $\vec{r}_2$ at the time t_2, with $t_2 > t_1$, provided that the distance between the two points $\leq c(t_2 - t_1)$.

Point 1) above is the more delicate because it involves the passage from the action-at-a-distance view to that of the field. The study by Zuza *et al.* [11], discussed in the Introduction, shows how this passage disorients university students of the first two years. We suggest that the traditional way to introduce the field presented by textbooks (see the second page of this paper) does not help this conceptual transition and that a new approach is necessary. In the next section, we shall see how a passage from Feynman's *Lectures* can help us. Point 2) leads to the introduction, from the beginning, of the idea that a point charge produces, in general, an electric and a magnetic field and that, coherently, the force exerted by these fields on a point charge is the Lorentz's force. Point 3) directly impacts the treatment of electromagnetic induction, the relation between the electric and the magnetic field during their propagation, and their causal connection with the sources (moving electric charges). Finally, we emphasize that the formulae are written as concisely as possible in the following sections. The teachers should adapt them to their teaching context.

2.1. The field concept

Introducing the concept of the field as a primary theoretical entity needs some practice of abstraction. As Feynman put it (see [7], p. 15–7):

> What we mean here by a field is this: a field is a mathematical function we use for avoiding the idea of action at a distance. If we have a charged particle at the position P, it is affected by other charges located at some distance from P. One way to describe the interaction is to say that the other charges make some "condition" —whatever it may be— in the environment at P. If we know that condition, which we describe by giving the electric and magnetic fields, then we can determine completely the behavior of the particle —with no further reference to how those conditions came about.
> [...]
> A field is then a set of numbers we specify in such a way that what happens *at a point* depends only on the numbers *at that point*. We do not need to know any more about what's going on at other places (3).

Feynman's conception of the electromagnetic field stresses that it is only a theoretical tool for describing electromagnetic phenomena, with no commitment to its existence in the world. Indeed, a theory aims to predict the values that the physical quantities can assume. Its aim is not to describe what is happening in the world: we do not have any means to ascertain that. This conception of the theories and their fundamental role contrasts textbooks' inductive and naively realistic stand.

According to point 1) above, the field concept must be introduced in the physics course as soon as possible. The occasion is, naturally, Newton's gravitational law. Besides the traditional equation in terms of the force of attraction between two (point) masses, teachers should introduce the description in terms of the gravitational field. A mass M produces at point $\vec{r}$ a gravitational field $\vec{g}$ given by

$$\vec{g} = -G\frac{M}{r^3}\vec{r}, \tag{5}$$

where G is the gravitational constant. The field $\vec{g}$ is such that, if another mass m is positioned at the point $\vec{r}$, then a force F given by

$$\vec{F} = m\vec{g}, \tag{6}$$

which acts on the mass m. Two points should be emphasized: A) the gravitational field has the dimensions of an acceleration and B) the conceptual scheme is: mass $\rightarrow$ field $\rightarrow$ force on another mass. Teachers should comment on how deeply the

(3) The original text speaks of a "real" field. We have omitted the adjective "real" because its use by Feynman concerns the epistemological status of the vector potential. Indeed, Feynman acknowledges from the beginning that "First we should say that the phrase 'a real field' is not very meaningful".

description in terms of field differs from that of the action-at-a-distance. Moreover, point A) suggests a series of reflections that could be developed according to the teaching context. See Appendix **A**.

Similarly, the treatment of Galilean-Newtonian relativity is the occasion of introducing the basic concepts of Einstein's relativity and discussing their fundamental differences. Teachers could do this based on the following points:

1) In an inertial reference frame, the acceleration measured by an accelerometer is null (see, for details, Appendix **A**). Accordingly, this is the best way of defining an inertial reference frame.
2) In both Galilean-Newtonian relativity and Einstein's, all phenomena develop in the same way in every inertial frame, *i.e.*, the equations describing each phenomenon have the same form in every inertial frame.
3) In both cases, the space is Euclidean, *i.e.*, it is homogeneous and isotropic, and the (variable) time is homogeneous.
4) The difference between the two approaches lies in that, while in the Galilean-Newtonian case, the coordinates' transformations are the so-called Galilean transformations, in Einstein's, the so-called Lorentz transformations are valid. The latter introduces the big novelty of a speed limit, given by the speed of light in a vacuum. This speed limit is responsible for the time-dilation and the length-contraction effects, as can be easily seen by putting in their formulae $c = \infty$, passing, in this way, from Lorentz's to Galilean-Newtonian transformations. In [13], chapter II, [14], chapter III, and [15], teachers will find derivations of the basic formulae of Einstein's kinematics obtained with thought experiments with the exchange of light pulses of ideal null duration between two inertial reference frames. The mathematics involved are elementary algebraic calculations.

2.2. The electromagnetic field produced by a moving charge

Treating electromagnetic phenomena within the conceptual framework of MLE requires a microscopic description of the phenomena. This description, in turn, implies that teachers must —as an introductory but fundamental part— give a picture of what matter is made of. How detailed this picture can be, depends primarily on the teaching context. This description should include information on what atoms and molecules are made of and how atoms enter and behave in conducting or insulating material. A focus should be put on the conducting mechanism in metals and the fact that electric currents in metals are made of moving electrons. Without ignoring that, by convention, mobile electric carriers are considered positive. The equation of the current density vector

$$\vec{J} = nq\vec{v}_q, \tag{7}$$

where n is the number of charges q per unit volume and $\vec{v}_q$ their velocity, should be written explicitly. From eq. (7), it follows that if the charge q is that of the electron, then the direction of the vector current density is opposed to that of the electrons' velocity. The electric current through a surface S is then defined as

$$I = -\int_S \vec{J}_e \cdot \hat{n} \, \mathrm{d}S, \tag{8}$$

where $\vec{J}_e$ is the electrons current density, $\hat{n}$ is the unit vector perpendicular to the surface element $\mathrm{d}S$, and where we have taken into account the convention about the current charge carriers. Instead of eq. (8), teachers can use the simplified version in which the surface S is perpendicular to the motion of the charges. This simplified treatment is particularly apt in the case of a metal wire. Considering this case, the stress must be on the charge velocity being the drift velocity.

Within this conceptual framework, teachers can state that, in general, an electric charge produces an electric field $\vec{E}$ and a magnetic field $\vec{B}$ and that an electromagnetic field exerts on a point charge q a force that is given by

$$\vec{F} = q\left(\vec{E} + \vec{v}_q \times \vec{B}\right), \tag{9}$$

where $\vec{v}_q$ is the velocity of the charge. Equation (9) is named "Lorentz force" ([4]).

These statements can be grounded on experimental observations. In a vacuum, the electric field produced at the point $\vec{r}$ by a charge *at rest* ($\vec{v}_q = 0$) at the origin has been proved to be

$$\vec{E} = \frac{1}{4\pi\varepsilon_0} q \frac{\vec{r}}{r^3}, \tag{10}$$

where ε_0 is the dielectric constant in a vacuum. Equation (10) has been corroborated by experiments with the Cavendish method [17]. Precisely, this method tests the formula

$$E = \frac{1}{4\pi\varepsilon_0} \frac{q}{r^{(2\pm\alpha)}}, \tag{11}$$

with the aim of reducig the value of α as much as possible. Cavendish obtained $\alpha \leq 2 \times 10^{-2}$; Maxwell improved to $\alpha \leq 1/21600 \approx 4.6 \times 10^{-5}$ (see [18], p. 77). Modern measurements have reduced the value of α to about 10^{-17} [19].

In discussing Cavendish's method, teachers should stress the relevance of continuously increasing the accuracy of our knowledge of fundamental physical laws and constants. They should also underline that it is based on an axiomatic approach. Indeed, the inverse square law is assumed to be true, and its implication —the inside conducting sphere must be free of charges— is tested by experiment. This axiomatic approach should be compared with Coulomb's inductive experiment to underlying the variety of methods used by physicists to unveil the properties of phenomena.

([4]) Indeed, as shown in [6, 16], Maxwell anticipated the expression of the Lorentz force.

As for the magnetic field, the issue is more delicate. We could begin by recalling Ørsted's experiment on the magnetic effect of a continuous current. Since macroscopic currents in metals are made of electrons moving with constant velocity (drift velocity), we can assume that a moving charge produces a magnetic field. Since the expression of the magnetic field can be obtained only by fully developing the implications of the modern formulation of Maxwell's equation in a vacuum, we can only state that the magnetic field produced by a moving charge is given by (5)

$$\vec{B} \approx \frac{\mu_0}{4\pi} q \frac{\vec{v}_q \times \vec{r}_{21}}{r_{21}^3}\,, \tag{12}$$

where $\vec{v}_q$ is the velocity of the charge q and $\vec{r}_{21} = \vec{r}_1 - \vec{r}_2$ is the vector pointing from the position $\vec{r}_2$ of the charge to the position $\vec{r}_1$ of the point in which the field is calculated. The sign $\approx$ reminds us that eq. (12) is approximately valid if the velocity of the charge $v_q \ll c$ and its variations are sufficiently slow to ignore the acceleration effects. Within this approximation, the retarded quantities of the charge q (position and velocity) can be replaced by the actual ones. The validity of eq. (12) rests on its experimental corroboration. Indeed, eq. (12) can be used to calculate the magnetic field produced by a continuous current flowing in a long enough rectilinear wire or the magnetic field produced by a continuous current flowing in a wire of arbitrary form (Biot-Savart's law). These macroscopic equations have been experimentally tested. Teachers could also add that —in the same approximations of eq. (12)— the electric field produced by a moving charge is

$$\vec{E} \approx \frac{q}{4\pi\varepsilon_0 r_{21}^3}\left(\vec{r}_{21} - r_{21}\frac{\vec{v}_q}{c}\right). \tag{13}$$

By using the basic relation

$$\vec{B} = \frac{1}{c}\left(\frac{1}{r_{21}^*}\vec{r}_{21}^{\,*} \times \vec{E}\right) \approx \frac{1}{c}\left(\frac{1}{r_{21}}\vec{r}_{21} \times \vec{E}\right), \tag{14}$$

where $\vec{r}_{21}^{\,*}$ is the retarded distance between the charge and the point in which the field is calculated, one can obtain the expression of the magnetic field (12). Going on, we should develop some order of magnitude calculations. Let us consider a long enough rectilinear metallic wire with a steady current. If the wire has a section of a square millimeter and a current of one A flows in it, the electron's drift velocity comes out to be $\approx 7.34 \times 10^{-5}\,\mathrm{m\,s^{-1}}$. Then, the second term of eq. (13) is approximately 2.44×10^{-13} smaller than Coulomb's term, and can be ignored in the calculation of the electric field produced by the electron. However, its presence is fundamental in calculating the magnetic field produced by a slowly moving electron (eq. (13)).

(5) For the calculation of the electromagnetic field produced by an arbitrarily moving charge, see, for instance, [20], pp. 870–877.

The teachers should adapt the above treatment to their teaching context by keeping the essential concept: a moving charge produces an electric and a magnetic field responsible for the magnetic effects of the current flowing in a wire. Here, there is an intriguing problem. We have stated that the magnetic field is produced by moving charges. Then, we learned that a moving charge adds a correction to the value of the electric field produced by the same charge at rest. Is there a physical quantity that can describe both phenomena? Teachers know that this quantity exists and is the so-called vector potential $\vec{A}$.

2.3. The vector potential

Teachers should say at least some words about the vector potential to illustrate the conceptual role played by it. Students are introduced from the beginning to the scalar potential φ. Then, the idea that another potential exists should not appear as a strange thing. The sources of the scalar potential are static distributions of charges; the sources of the vector potential are the currents, namely, charges in motion. From the knowledge of its sources, *i.e.*, charges in motion, we can calculate the value of the vector potential $\vec{A}$. Knowing the vector potential, we can calculate the magnetic field produced by the moving charges through a relation involving a particular spatial variation of the vector potential. Teachers know that this relation is $\mathrm{curl}\,\vec{A} = \vec{B}$ (it is not necessary to show this equation to the students). Moreover, a particular temporal variation of the vector potential yields the contribution of the moving charges to the electric field ($-\partial\vec{A}/\partial t = \vec{E}$). Then the complete expression of the electric field is given by the sum of the contributions from charges at rest and from charges in motion: $\vec{E} = -\,\mathrm{grad}\,\varphi - \partial\vec{A}/\partial t$. Again, it is not necessary to show this equation to the students. In [6], sec. VII, the reader will find a detailed proposal for introducing the vector potential in elementary physics and high school courses. A less recent proposal to introduce the vector potential in high schools can be found in [21].

2.4. Electromagnetic induction

Textbooks and teaching practices describe electromagnetic induction with what Feynman labeled as the "flux rule", downgrading it from the status of physical law (see [7], pp. 17.1–17.3). The "flux rule" states that

$$\mathcal{E} = -\frac{\mathrm{d}}{\mathrm{d}t}\int_S \vec{B}\cdot\hat{n}\,\mathrm{d}S = -\frac{\mathrm{d}\Phi}{\mathrm{d}t}\,, \tag{15}$$

where $\vec{B}$ is the magnetic field, and S is any surface that has the circuit as a contour. As shown in [6], the "flux rule" is not a physical law but only a calculation shortcut that must be handled carefully. Instead, the law of electromagnetic induction is founded on the definition of the induced *emf* as [5, 6]

$$\mathcal{E} = \oint_l \left(\vec{E} + \vec{v}_c\times\vec{B}\right)\cdot\mathrm{d}\vec{l} = \oint_l \vec{E}\cdot\mathrm{d}\vec{l} + \oint_l \left(\vec{v}_c\times\vec{B}\right)\cdot\mathrm{d}\vec{l}, \tag{16}$$

where the electric field $\vec{E}$ and the magnetic field $\vec{B}$ are solutions of Maxwell's equation, and $\vec{v}_c$ is the velocity of the positive charges that, by convention, are the current carriers. This integral yields —numerically— the work done by the electromagnetic field on a unit positive charge through the entire loop. Equation (16) is local because it connects the physical quantity $\mathcal{E}$ defined on the line l at the time t to other physical quantities defined at every point of the line l at the same instant t.

The expression of the electric field in eq. (16) contains a particular time dependence of the vector potential $\vec{A}$ (its partial derivative with respect to time $-\partial\vec{A}/\partial t$), as explained in sect. 2.3. Then, the induced *emf* is the sum of two line integrals as shown by the last equality of eq. (16). The induced *emf* thus obtained describes all known phenomena of electromagnetic induction [6]. See also Appendix **C**.

Let us apply eq. (16) to the relative inertial motion of a magnet and a rigid, filiform circuit. For Einstein, this (thought) experiment was one of the reasons for founding special relativity (see [22], Engl. transl., p. 140). In the reference frame of the magnet, there is no electric field. Therefore, only the second integral of eq. (16) is operative. Instead, in the reference frame of the circuit, both integrals are, in principle, operative. However, the last integral of eq. (16) is null because $\vec{v}_c = \vec{v}_d$ and $\vec{v}_d$ is always parallel to $\mathrm{d}\vec{l}$. The circuit sees the vector potential produced by the magnet varying with time owing to the relative motion between the magnet and the circuit. In conclusion: in the reference frame of the magnet, only the magnetic component of Lorentz's force on a unit positive charge is operative; in the reference frame of the circuit, only the time variation of the vector potential operates.

It will be of great pedagogical value to experiment on this fundamental topic. An experiment of this kind has been described in detail in [23]. The laboratory session is held before any electromagnetic induction discourse but after the special relativity lessons. Students, divided into pairs, are invited to experiment at will. After about an hour or so of experimenting, students are asked to describe what they have seen with a formula. The teacher intervenes as little as possible. Spontaneously, the students describe the observed phenomena in the magnet reference frame. Then, the students are asked to describe their observations in the reference frame of the moving coil. After some discussion, the teacher suggests to guess a formula that obeys the locality principle. In this way, students learn or apply the principle that the equation describing a phenomenon must have the same form in every inertial frame.

It is possible to rewrite eq. (16) in terms of a single surface integral under severely restricting conditions concerning the integral (6):

$$\oint_l \left(\vec{v}_c \times \vec{B}\right) \cdot \mathrm{d}\vec{l}. \tag{17}$$

Let us consider a rigid and filiform circuit that moves with velocity V in the laboratory. Let us further assume that the motion of the circuit occurs along the positive direction

(6) The following calculations are for the teachers. Considering the available mathematical tools, they should adapt them to their teaching context.

of the x axis. In the Galilean limit ($c = \infty$), the velocity of the charge $\vec{v}_c$ can be written as $\vec{v}_c = \vec{V} + \vec{v}_d$, where $\vec{v}_d$ is the drift velocity of the charges ([7]). Then, eq. (16) assumes the form

$$\mathcal{E} = \oint_l \vec{E} \cdot \mathrm{d}\vec{l} + \oint_l \left(\vec{V} \times \vec{B}\right) \cdot \mathrm{d}\vec{l} + \oint_l \left(\vec{v}_d \times \vec{B}\right) \cdot \mathrm{d}\vec{l}, \tag{18}$$

where all the line integrals are evaluated in the laboratory reference frame. After some calculations [6], it can be proved that the induced *emf* is given by

$$\mathcal{E} = -\frac{\mathrm{d}}{\mathrm{d}t} \int_S \vec{B} \cdot \hat{n} \, \mathrm{d}S + \oint_l \left(\vec{v}_d \times \vec{B}\right) \cdot \mathrm{d}\vec{l}. \tag{19}$$

The line integral is null for filiform circuits because the drift velocity $\vec{v}_d$ is always parallel to $\mathrm{d}\vec{l}$. Then, we get the "flux rule". This rule has been obtained in the Galilean limit and for inertially moving rigid and filiform circuits. Equation (19) is also valid in the reference frame of the circuit. Indeed, the "flux rule" is Galileo-invariant, as can be easily proved. In the Galilean limit $\vec{B}' = \vec{B}$, $t' = t$, and $S' = S$, where the primed quantities refer to the circuit reference frame. Then $\mathrm{d}\Phi'/\mathrm{d}t' = \mathrm{d}\Phi/\mathrm{d}t$.

The "flux rule" is a piece of Galilean-Newtonian physics within the Lorentz invariant theory of MLE. Approximations in the Galilean-Newtonian limit can, of course, be used. However, an inescapable condition is to discuss with the students the serious (physical and epistemological) problems posed by the "flux rule". Moreover, the Galilean limit of the law of electromagnetic induction is conceptually very different from the Newtonian limit of relativistic dynamics. While Newtonian dynamics can be interpreted causally, the "flux rule" cannot (see below).

Therefore, teachers should underline that:

– The "flux rule" implies an improper use of the field concept, because it describes what is going on in the closed circuit with what happens —at the same instant— on an arbitrary surface with the circuit as a contour. In this way, the essential feature of the field concept is lost: a field is a set of numbers we specify in such a way that what happens at a point *of the circuit* depends only on the numbers *at that point*. We do not need to know anymore about what is happening at other places (*on the surface with the circuit as a contour*). The reader will recognize in this statement what Feynman said in the quote in sect. 2.1, adapted to our case.

([7]) For a rigid and filiform circuit at rest in the laboratory, the drift velocity $\vec{v}_d$ is defined as the velocity of mobile charges when a steady or slowly varying current flows. When the circuit moves inertially with velocity V along the positive direction of the common $x \equiv x'$ axis, the drift velocity is defined in the moving reference frame in which the circuit is at rest and is denoted by $\vec{v}'_d$. From the above definition, it follows that $\vec{v}_d = \vec{v}'_d$, because every phenomenon develops similarly in every inertial frame. In other words: if we measure the drift velocity in a circuit in the laboratory, we shall find a specific value q. If the *same* circuit is in the moving inertial frame, and we measure the drift velocity in this frame, we shall find the exact value q measured in the laboratory. Of course, this equivalence is true in special and Galilean relativity.

– The "flux rule" cannot be interpreted causally because it relates the physical quantity $\mathcal{E}$ defined on the line l at the instant t to the values of the magnetic field $\vec{B}$ defined at all points of an arbitrary surface at the same time t, thus implying the propagation of physical interaction with infinite speed (see also the discussion of eq. (21) in the next section).
– It cannot say where the induced *emf* is localized [6]. To illustrate this point, teachers should discuss the case (generally treated in textbooks) of a bar sliding on a U-shaped conductive frame immersed in a constant and uniform magnetic field. As shown in [6], the induced *emf* is localized in the bar for both inertial reference frames (the laboratory's and the bar's).
– Frequently, it requires an *ad hoc* choice of the path used as a contour of the integration surface [24].
– As shown by Blondel (1914) [25], it is falsified by a clear-cut experiment [6].

In a study of electromagnetic induction understanding by first years university students, Guisasola *et al.* found "that most of the students failed to distinguish between macroscopic levels described in terms of fields and microscopic levels described in terms of the actions of fields" [26]. According to the authors, the "flux rule" is a macroscopic description, while Lorentz's force is microscopic. The definition of the induced *emf* given by eq. (16) is a microscopic description. If developed coherently, it leads to a microscopic theory of electromagnetic induction.

Teaching electromagnetic induction with a full microscopic description will avoid the use and the pitfalls of the macroscopic description of the "flux rule". Teachers have to make a choice depending on their teaching context. If, as tradition, textbooks, and teaching habits imply, the choice is the "flux rule", this choice should be accompanied by a full discussion of its physical and epistemological drawbacks. What should be avoided is speaking of the "flux rule" as the law of electromagnetic induction without any critical discussion, which, by the way, would stimulate the students' critical reasoning.

2.5. What to say about Maxwell's equations?

A choice is that of ignoring them. Giancoli does not even mention Maxwell [4]. Another option is to write them in integral form. Italian textbooks for high school widely adopt this choice (see [27], pp. 233–234, [28], pp. 299–306, [29], pp. 105–106). In the United States, we have encountered an example in Halliday, Resnick, and Walker's book (see [3], pp. 941–951). Instead, Cutnell and Johnson do not mention Maxwell's equations [30] ([8]).

([8]) We have not been able to ascertain if these texts are considered in the United States or in other English speaking countries as textbooks for high school or for higher teaching levels. We know that their Italian translations are considered as textbooks for high schools.

Indeed, all textbooks speak about two of Maxwell's equations in integral form (or as a sum of finite terms of the type $\vec{E} \cdot \vec{\Delta S}$ (Gauss law (9)) or $\vec{B} \cdot \vec{\Delta S}$ ("flux rule"), perhaps without labeling them as Maxwell's equations.

Writing Maxwell's equations in integral form is conceptually deceptive. For instance, consider the equation

$$\operatorname{curl} \vec{E} = -\frac{\partial \vec{B}}{\partial t}\,, \tag{20}$$

and its integral form

$$\oint_l \vec{E} \cdot \mathrm{d}\vec{l} = -\frac{\mathrm{d}}{\mathrm{d}t} \int_S \vec{B} \cdot \hat{n}\, \mathrm{d}S. \tag{21}$$

This equation relates what happens on the closed line l at the time t to what happens, at the same time t, on an arbitrary surface S with the line l as a contour. This equation cannot be interpreted causally because physical interactions cannot propagate at infinite speed. Equation (21) only establishes a relation between quantities defined on the line with quantities on the arbitrary surface chosen.

Textbooks and teaching practices, in discussing eq. (21), state that a varying magnetic field produces (causes) an electric field; and, conversely, from the equation of the curl of the magnetic field (or its integral form), they state that a varying electric field produces (causes) a magnetic field. These statements are untenable because the electric and magnetic fields are produced (caused) by charges in motion. Therefore, the equations of the curl of the electric and magnetic fields (or their integral form) only establish a relation between these fields without any causal connection between them. These issues are widely discussed in [6, 31].

Teachers face a crossroads. Keeping on using Maxwell's equations in integral form, explicitly (as Italian teachers, following their textbooks, do) or follow a more challenging way outlined in Appendix **B**. In the first case, teachers should explain the physical and epistemological drawbacks of this choice.

3. Discussion and conclusions

Teachers' resistance to proposed changes in teaching coming from central or local institutions is well-studied in the literature. Powell and Kusuma-Powell distinguish between "technical" and "adaptive" changes. Technical changes require informational learning. Instead, adaptive changes "call for transformational learning or learning that requires us to rethink our deeply held values, beliefs, assumptions, and even our

(9) Coulomb's law is written in terms of forces between point charges; instead Gauss's law is written in terms of the electric field, *i.e.*, of a quantity defined at every point of the considered surface. If this step is done without a clear explanation, it would likely confuse the students because it overlaps the action-at-a-distance and the field descriptions.

professional identity. Adaptive challenges are complex, and addressing them requires patience and time" (see [32], p. 67). Our proposal demands teachers the disposition to:

a) abandon the centuries-old tradition of presenting electromagnetic phenomena following their chronological development;
b) leave behind the epistemological stand according to which physical laws must be induced only from experiment;
c) acknowledge the fundamental role played by the abstraction and the hypothetical-deductive method;
d) recognize the necessity of some essential, microscopic descriptions.

These features place our proposal in the "adaptive changes" category and could be evaluated as too radical to be implemented by teachers. We have presented an early version of our proposal to a group of about twenty-five Italian teachers we meet periodically online. The majority of these teachers teach in scientific high schools. Positive reactions came from retired teachers. The negative reaction has been substantially based on the following:

1) The backwardness of the teaching context.
2) The constraints of the programs of the Ministry of Education and the local tendency to standardize the teaching practices in all the classes, with the adoption of the same textbooks, also in the view of preparing the students for the final state exam.
3) The non-necessity of teaching MLE Electromagnetism; some Galilean approximation of standard courses is sufficient.
4) The mathematical difficulties.

The first two points are sadly founded. As explained above, we cannot agree with point 3) because we believe that electromagnetic phenomena must be taught within the conceptual framework of MLE electromagnetism. Instead, we have thoroughly considered the last point 4). Meanwhile, four teachers in our group have agreed to the project of experimenting with the present proposal in their classes. We are actively working with them on this project with about a meeting per month.

* * *

We warmly thank Maria Grazia Blumetti, Elena Failla, Andrea Farusi, and Marco Litterio for their suggestions and commitment to experiment with this project.

Appendix

In this Appendix, teachers will find some development of topics in the main text that could be used in favorable teaching conditions or as teachers' background knowledge. For instance, while Appendix **A** belongs to the former group, the other two sections are primarily —but not exclusively— intended for the teachers' background knowledge.

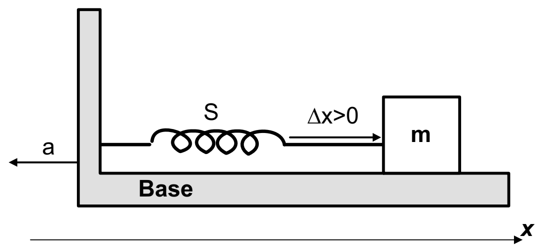

Fig. 1. – Working principle of an accelerometer.

A. *Gravitational field* – As we have seen in sect. 2.1 of the main text, the gravitational field has the dimensions of an acceleration. Accelerations can be measured with an accelerometer in the accelerated reference frame (fig. 1).

The mass m is connected to a rigid base by the spring S; ideally, it can slide on the base without friction. Suppose the base is subjected to a constant acceleration to the left. In that case, the spring is stretched, and its maximum extension Δx is related to the acceleration of the base by the equation

$$\overrightarrow{\Delta x} = -\frac{m}{k}\vec{a}, \tag{A.1}$$

where k is the spring's constant. The elongation of the spring occurs along the opposite direction of the base acceleration. If the accelerometer is rotated 90 degrees to the right, it will find itself in the vertical position. The spring elongates towards the ground, owing to the effect of the gravitational field $\vec{g}$ on the mass m: the accelerometer becomes a gravimeter. In this case, the accelerometer indicates an acceleration equal to $-\vec{g}$ directed upwards (10).

If a laboratory —with the accelerometer fixed in the vertical position on a wall— is free falling in a gravitational field, the spring does not elongate because the (acceleration) field $-\vec{g}$ is canceled out by the acceleration $\vec{g}$ due to free fall: the measured acceleration is null. Since we have defined an inertial reference frame as the one in which the measured acceleration is null, it follows that a free-falling laboratory constitutes an inertial reference frame. Moreover, this thought experiment suggests that the effect of a gravitational field $\vec{g}$ on a mass m is equivalent to the effect of an acceleration field $-\vec{g}$ on the same mass. Therefore, we can conclude that $m_g = m_i$, where m_g is the "gravitational mass" and m_i is the "inertial mass" which appears in the Newtonian equation $\vec{F} = m_i\vec{a}$. In metric theories of gravitation, this property is

(10) The gravitational field measured on the Earth surface depends on the latitude, also if we suppose that the Earth surface is spherical. In fact, in the accelerated reference system centered at the Earth center and rotating with the Earth, the component of the centripetal acceleration perpendicular to the Earth surface is equivalent to a pseudo-gravitational field directed upwards. This pseudo-gravitational field decreases the value of g measured by an accelerometer. In particular, the gravitational field is smaller at the Equator than at the pole, as can be easily proved.

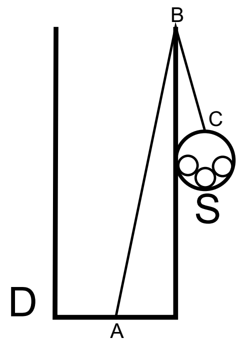

Fig. 2. – D is a plastic cylinder; A-B-C is a narrow band of elastic rubber; S is a plastic ball or cylinder (made of two parts that can be separated) containing a suitable number of coins. This device can be quickly built using materials easily found at home.

assumed as the "weak equivalence principle" ([11]). Since a the free-falling laboratory is an inertial reference frame, a body that is left free will remain at rest or, if endowed with an initial linear momentum, it will keep moving uniformly along a straight line. However, this is true only if the gravitational field is uniform: in general, gravitational fields are not. Hence, the previous statement is approximately verified, provided the laboratory sizes are sufficiently small.

The qualitative features of a free-falling body can be demonstrated in the classroom using the simple device shown in fig. 2 ([12]). The teacher should perform two experiments. Before doing each experiment, the teacher illustrates what he will do and asks the student what will happen. The first experiment lets the device fall from the teacher's hand and is positioned from the ground at the highest possible level. The ball containing the coins will be retracted into the cylinder during free fall. The second experiment consists in launching the cylinder toward the ceiling. The ball containing the coins will be retracted into the cylinder already during the ascent to the ceiling, thus demonstrating that the free fall is the motion of a mass subjected only to a gravitational field. The discussion with the student will also encompass the negligible effect of the atmosphere.

Finally, it would be interesting to discuss with the students a passage from a book by Galileo Galilei that reads (see [33], pp. 63-64):

> A large stone placed in a balance not only acquires additional weight by having another stone placed upon it, but even by the addition of a handful of hemp its weight is augmented six to ten ounces according to the quantity of hemp. But if you tie the hemp to the stone and allow them to fall freely from some height, do you believe that the hemp will press down upon the stone and thus

([11]) In special relativity, the mass m is no longer a measure of a body inertia. Indeed, the concept of inertial mass rests on using the equation $\vec{F} = m\vec{a}$, which is no longer valid in special relativity.

([12]) This home-made device has been suggested to one of the author (G.G.) by Prof. Mauro Carfora.

accelerate its motion or do you think the motion will be retarded by a partial upward pressure? One always feels the pressure upon his shoulders when he prevents the motion of a load resting upon him; but if one descends just as rapidly as the load would fall how can it gravitate or press upon him? Do you not see that this would be the same as trying to strike a man with a lance when he is running away from you with a speed which is equal to, or even greater, than that with which you are following him? You must therefore conclude that, during free and natural fall, the small stone does not press upon the larger and consequently does not increase its weight as it does when at rest ([13]).

B. *Maxwell's equations* – In sect. 2.5 of the main text, we have discussed using (at least two) Maxwell's equations written in integral form: we have stressed this treatment's physical and epistemological drawbacks. Generally speaking, the teaching contexts allow a variety of choices. In the following, we will outline a more challenging path that could —perhaps— be followed in suitable conditions.

Teachers could write Maxwell's equations in a vacuum in their differential form without, however, specifying the expression of the divergence and curl operators:

$$\operatorname{div}\vec{E} = \frac{\rho}{\varepsilon_0}\,, \tag{B.2}$$

$$\operatorname{curl}\vec{E} = -\frac{\partial\vec{B}}{\partial t}\,, \tag{B.3}$$

$$\operatorname{div}\vec{B} = 0, \tag{B.4}$$

$$\operatorname{curl}\vec{B} = \mu_0\left(\vec{J} + \varepsilon_0\frac{\partial\vec{E}}{\partial t}\right), \tag{B.5}$$

and comment on them in the following way:

1) The operator divergence and curl operate on the spatial variations of the vector to which they are applied.
2) These equations relate the *sources* ρ (charge density) and $\vec{J}$ (current density) to the electric field $\vec{E}$ and to the magnetic field $\vec{B}$.

([13]) Una gran pietra messa nella bilancia non solamente acquista peso maggiore col soprapporgli un'altra pietra, ma anco la giunta di un pennecchio di stoppa la farà pesar più quelle sei o dieci once che peserà la stoppa; ma se voi lascerete liberamente cader da un'altezza la pietra legata con la stoppa, credete voi che nel moto la stoppa graviti sopra la pietra, onde gli debba accelerar il suo moto, o pur credete che ella la ritarderà, sostenendola in parte? Sentiamo gravitarci su le spalle mentre vogliamo opporci al moto che farebbe quel peso che ci sta addosso; ma se noi scendessimo con quella velocità che quel tal grave naturalmente scenderebbe, in che modo volete che ci prema e graviti sopra? Non vedete che questo sarebbe un voler ferir con la lancia colui che vi corre innanzi con tanta velocità, con quanta o con maggiore di quella con la quale voi lo seguite? Concludete pertanto che nella libera e naturale caduta la minor pietra non gravita sopra la maggiore, ed in consequenza non le accresce peso, come fa nella quiete (see [34], pp. 76-77).

3) The first equation (B.2) states that the operator divergence applied to the electric field $\vec{E}$ yields ρ/ε_0.
4) The third equation (B.4) says that the divergence of the magnetic field is always null. This result implies that the magnetic field has no sources similar to the charge density for the electric field. Indeed, the magnetic field sources are the currents densities, *i.e.*, charges in motion.
5) The second equation (B.3) connects spatial variations of the electric field $\vec{E}$ to the time variation of the magnetic field $\vec{B}$.
6) The fourth equation (B.5) connects spatial variations of the magnetic field to its source $\vec{J}$ and to the time variation of the electric field.
7) Given the sources ρ and $\vec{J}$, the physical dimensions of the electric and magnetic fields remain undeterminate, together with those of the two constants ε_0 and μ_0.
8) The assumption of the Lorentz force $\vec{F} = q(\vec{E} + \vec{v}_q \times \vec{B})$ gives physical dimensions to the two fields and the two constants.
9) The value of the two constants ε_0 and μ_0 must be determined experimentally.
10) Maxwell's equations (B.2)–(B.5) describe all electromagnetic phenomena observed in a vacuum. Further assumptions must be made for describing electromagnetic phenomena in a material.
11) The solutions of Maxwell's equations describe how electromagnetic signals produced by the sources propagate. In a vacuum, their propagation velocity is $c = 1/\sqrt{\varepsilon_0 \mu_0}$.
12) If the sources do not depend on time, Maxwell's equations describe electrostatic phenomena.
13) In 1888, Hertz demonstrated that electromagnetic waves reflect, refract, and diffract as light waves; they are also polarized. Light and electromagnetic waves obey the same equations. Hence, light can be described as an electromagnetic wave.
14) Special relativity shows that c is a limit speed.
15) As for the two constants, their numerical values are obtained by putting $\mu_0 = 4\pi \times 10^{-7}\,\mathrm{N\,A^{-2}}$ and deducing ε_0 from the formula yielding the light velocity in a vacuum determined experimentally.
16) Teachers should add that Maxwell's equations written for a magnetic material assume that their magnetic properties are due to currents circulating in the material (Ampère's currents). Indeed, a sound explanation of magnetic properties requires a quantum mechanical treatment.

C. *Electromagnetic induction* – In sect. 2.4 of the main text, we have seen how the "flux rule" is only a calculation shortcut and pointed out that the law of electromagnetic induction is founded on the definition of the induced *emf* as

$$\mathcal{E} = \oint_l \left(\vec{E} + \vec{v}_c \times \vec{B}\right) \cdot \mathrm{d}\vec{l} = \oint_l \vec{E} \cdot \mathrm{d}\vec{l} + \oint_l \left(\vec{v}_c \times \vec{B}\right) \cdot \mathrm{d}\vec{l}. \tag{C.6}$$

We have observed that this equation is local and that also its solution must be local. Consequently, both equations can be interpreted causally. In the following, we develop some calculations that should be part of the background knowledge of teachers on this topic.

Within the description of MLE in terms of the electromagnetic potentials, the general expression of the electric field is given by

$$\vec{E} = -\nabla\varphi - \frac{\partial \vec{A}}{\partial t}\,, \tag{C.7}$$

where φ and $\vec{A}$ are the scalar and the vector potential. Consequently, eq. (C.6) assumes the form

$$\mathcal{E} = \oint \left[\left(-\nabla\varphi - \frac{\partial \vec{A}}{\partial t}\right) + \left(\vec{v}_c \times \vec{B}\right)\right] \cdot \mathrm{d}\vec{l} = \oint_l \left[\left(-\frac{\partial \vec{A}}{\partial t}\right) + \left(\vec{v}_c \times \vec{B}\right)\right] \cdot \mathrm{d}\vec{l}, \tag{C.8}$$

because the line integral $\oint_l \mathrm{grad}\,\varphi \cdot \mathrm{d}\vec{l} = 0$.

If we want to get the "flux rule", we must start again from eq. (C.6), and write, in the reference frame of the laboratory

$$\begin{aligned}\mathcal{E} = \oint_l \vec{E} \cdot \mathrm{d}\vec{l} + \oint_l \left(\vec{v}_c \times \vec{B}\right) \cdot \mathrm{d}\vec{l} &= \int_S \mathrm{curl}\,\vec{E} \cdot \hat{n}\,\mathrm{d}S + \oint_l \left(\vec{v}_c \times \vec{B}\right) \cdot \mathrm{d}\vec{l} \\ &= -\int_S \frac{\partial \vec{B}}{\partial t} \cdot \hat{n}\,\mathrm{d}S + \oint_l \left(\vec{v}_c \times \vec{B}\right) \cdot \mathrm{d}\vec{l},\end{aligned} \tag{C.9}$$

where S is any *arbitrary* surface that has the integration line l as a contour. For every vector field with mull divergence (see [20], pp. 10-11)

$$\int_S \frac{\partial \vec{B}}{\partial t} \cdot \hat{n}\,\mathrm{d}S = \frac{\mathrm{d}}{\mathrm{d}t}\int_S \vec{B} \cdot \hat{n}\,\mathrm{d}S + \oint_l \left(\vec{v}_l \times \vec{B}\right) \cdot \mathrm{d}\vec{l}, \tag{C.10}$$

where $\vec{v}_l$, the velocity of the line element $\mathrm{d}l$, can be different for each line element. Therefore, eq. (C.9) becomes

$$\mathcal{E} = -\frac{\mathrm{d}\Phi}{\mathrm{d}t} - \oint_l \left(\vec{v}_l \times \vec{B}\right) \cdot \mathrm{d}\vec{l} + \oint_l \left(\vec{v}_c \times \vec{B}\right) \cdot \mathrm{d}\vec{l}. \tag{C.11}$$

In the case of a *rigid, filiform circuit* moving with velocity V along the positive direction of the common $x' \equiv x$ axis, this equation becomes

$$\mathcal{E} = -\frac{\mathrm{d}\Phi}{\mathrm{d}t} - \oint_l \left(\vec{V} \times \vec{B}\right) \cdot \mathrm{d}\vec{l} + \oint_l \left(\vec{v}_c \times \vec{B}\right) \cdot \mathrm{d}\vec{l}. \tag{C.12}$$

We can write $\vec{v}_c = \vec{V} + \vec{v}_d$ in the Galilean limit ($c = \infty$). Then, finally

$$\mathcal{E} = -\frac{\mathrm{d}\Phi}{\mathrm{d}t} + \oint_l \left(\vec{v}_d \times \vec{B}\right) \cdot \mathrm{d}\vec{l} = -\frac{\mathrm{d}\Phi}{\mathrm{d}t}\,, \tag{C.13}$$

i.e., the "flux rule" (the line integral is null because for every line element, $\vec{v}_d$ is parallel to $\mathrm{d}\vec{l}$).

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