Conservation laws and laser cooling of atoms

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Abstract.

The straightforward application of energy and linear momentum conservation to the absorption/emission of photons by atoms allows to establish the essential features of laser cooling of two levels atoms at low laser intensities. The lowest attainable average kinetic energy of the atoms depends on the ratio Γ/E_R between the natural linewidth and the recoil energy and tends to E_R as Γ/E_R tends to zero (in one dimension). This treatment, like the quantum mechanical ones, is valid for any value of the ratio Γ/E_R and contains the semiclassical theory of laser cooling as the limiting case in which $E_R \ll \Gamma$.

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1 Introduction

Starting from the pioneering works of mid Seventies of last century, laser cooling of atoms has become a vast research field with many applications in physics, chemistry and biology.

When an atom absorbs a photon, its kinetic energy is changed according to the laws of energy and linear momentum conservation. In general, but not always, if an atom flying against a photon absorbs it, the kinetic energy of the atom decreases. Instead, if a photon is chasing the atom, the variation of the atom's kinetic energy due to the absorption of the photon is always positive.

The proposal of cooling atoms with laser beams has been put forward by Hänsch and Schawlow in 1975 [1]: the idea was that of illuminating the atoms with six laser beams (two opposite beams for each spatial dimension) red detuned with respect to an absorption line.

Apparently, there are two limits to the cooling process: the Doppler limit and the recoil limit. The Doppler limit is due to the natural width $\Gamma = \hbar\gamma$

(Full Width at Half Maximum) of the atomic transition used: since the first order Doppler shift is $\pm\Delta E(v/c)$ (where ΔE is the transition energy between the two atomic levels), when $\Delta E(v/c) \approx \Gamma$, the photon may be absorbed (with significant probability) not only by atoms flying against the photon but also by atoms flying in the opposite direction, thus limiting the cooling process. Instead, the recoil limit is due to the fact that when the kinetic energy of the absorbing atom is of the same order of magnitude of its variation due to the absorption or emission of a photon no further cooling seems to be possible.

The theoretical treatment of laser cooling is not a simple one. In the semi–classical approach, the atom is considered as a localized two–levels quantum system and the light field is treated classically [2]. According to this theory, the lowest attainable average kinetic energy $< E_K >$ of the atoms is given by $\Gamma/4$. This result can not be valid in all conditions since it implies that $< E_K > \to 0$ as $\Gamma \to 0$. This physically unsound result is due to the assumption that the photon momentum is negligible with respect to the atomic one. This approximation implies that the kinetic energy of the atom is much larger than the recoil energy E_R. As we shall see, the semiclassical theory is applicable only when $\Gamma \gg E_R$.

A quantum mechanical treatment of the motion of a two levels atom under laser light, applicable for any value of the ratio between the natural linewidth and the recoil energy, has been developed, among others, by Winelnd and Itano [3] and by Castin, Wallis and Dalibard [4].

The discovery by Lett et al. that temperatures well below the Doppler limit can be achieved [5] inspired a re–formulation of the theoretical description: it was found that the multilevel nature of the atoms and the spatial variation of the light field polarization can be exploited for attaining temperatures close to the recoil limit [6].

Temperatures below the recoil limit can be achieved by sophisticated procedures like the Velocity Selective Coherent Population Trapping [7] and the Stimulated Raman Cooling [8]. The book by Metcalf and van der Straten may be taken as a guide for exploring all these issues and the vast available literature [9].

Stimulated by these developments, the search for ever lower temperatures has attracted the attention of experimental and theoretical physicists. This notwithstanding, the theoretical reconsideration of laser cooling of two levels atoms maintains its importance, since it sheds light on essential features of the cooling process.

This paper is based on the idea that the application of conservation laws to the absorption/emission of photons by atoms should yield the essential features of laser cooling of two levels atoms at low laser intensities.

The conceptual framework of the following treatment is simple and the mathematics needed is limited to some algebraic manipulations. Therefore, this paper might be of some interest for university and high school teaching or for in service training of high school teachers. It might also be of some value for researchers: being based on conservation principles, it sets down limiting conditions that should be met also by more sophisticated (and complicated) treatments like the quantum mechanical ones.

The first treatment of the absorption/emission of photons by atoms based on energy and linear momentum conservations is due to Schrödinger' [10]. Schrödinger's paper has been ignored by his contemporaries and even nowadays it is not so popular: see, for instance, [11, 12, 13]. Reasonably, this oblivion has been due to the deep rooting of the wave description of light in the background physical knowledge, in spite of the fact that the absorption/emission of light by atoms is a discrete process. Schrödinger's treatment is a relativistic one. We shall maintain this approach, notwithstanding the fact that the atoms' velocities of interest in the final steps of the laser cooling process are much smaller than the light speed. This choice is motivated by several reasons. Firstly because Schrödinger's relativistic treatment can be applied to a variety of phenomena ranging from the emission of electromagnetic radiation by atoms/nuclei in inertial flight at relativistic velocities [12] to the emission/absorption of photons without recoil (Mössbauer effect) on rotating devices [13]; more fittingly to the subject of the present paper, because it could be applied to laser cooling of ions at relativistic speeds in storage rings, a research field initiated by Schröder et al. in 1990 [14]. Secondly, because by using directly the approximation of newtonian mechanics the atom's rest energy, which appears in the expression of the recoil energy, remains conceptually obscure. In A it is shown how to obtain the basic approximated formulas by applying directly the newtonian mechanics: the teacher's choice will depend on the students' level, on epistemological considerations and on personal preferences. Instead, in B the exact relativistic formulas are applied to ions in storage rings.

2 Outline of the paper

An outline of the main steps of the paper will help in understanding how the various calculations contribute to the overall description. We shall deal only with energies: the transition energy ΔE_M, the recoil energy $E_R = \Delta E_M^2/2Mc^2$, the natural linewidth $\Gamma = \hbar\gamma$ and the energy E_{ph} of the absorbed/emitted photon. The transition energy ΔE_M is the one corresponding to the most probable value given by the Lorentzian shape of the natural line.

In section 3, we shall assemble the basic formulas, giving them the most suitable form for the laser cooling process. Two dimensionless parameters are introduced:

$$B_T = \frac{E_R}{\Delta E_M} = \frac{\Delta E_M}{2Mc^2} \tag{1}$$

and

$$B_D = \frac{\Gamma}{\Delta E_M} \tag{2}$$

Their ratio B_D/B_T is equal to Γ/E_R. A third dimensionless parameter will enter the description: the parameter $B_1 = v_1/c$, where v_1 is the norm of the atom's velocity vector $\vec{v}_1$ before the absorption/emission of a photon. The use of dimensionless parameters simplify the calculations and allows an easy comparison of the orders of magnitude involved. In order to accustom himself to the use of these parameters, the reader should keep in mind that B_T is a measure of the recoil energy, Γ of the natural linewidth in units of the transition energy ΔE_M and B_1 of the atom's velocity in units of c. Of course, when needed, the basic physical quantities will be re–established or shown along with the dimensionless parameters. The following table contains the data relative to typical atomic transitions used in laser cooling.
In section 4, the implications of a non zero linewidth are discussed. In particular, it is shown that when an atom absorbs a counter–propagating photon its transition energy is ΔE^A, while when the atom absorbs a co–propagating photon of the same energy its transition energy is $\Delta E^P = \Delta E^A(1 - B_1)/(1 + B_1)$.

In section 5, the basic formulas are approximated in the limit of small atoms' velocities: in this limit, only the first order Doppler effect is taken into account and the relativistic dynamics can be replaced by the Newtonian one. In order to identify this limit without ambiguity, we must take into account that the formulas contain terms of the first or higher order in the

Atom	λ [nm]	$B_T = E_R/\Delta E_M$	$B_D = \Gamma/\Delta E_M$	$B_D/B_T = \Gamma/E_R$	B_1 [T=300 K]
^{4}He	388.98	4.27×10^{-10}	1.93×10^{-9}	**4.52**	4.56×10^{-6}
^{7}Li	670.96	1.41×10^{-10}	1.32×10^{-8}	**93.62**	3.44×10^{-6}
^{23}Na	589.16	4.91×10^{-11}	1.97×10^{-8}	**401.22**	1.90×10^{-6}
^{133}Cs	852.35	5.87×10^{-12}	1.47×10^{-8}	**2504**	7.91×10^{-7}

Table 1: Typical values of the dimensionless parameters used in this paper. The parameter B_1 has been calculated by putting $B_1 = v_{rms}/c$ at 300 K. For reasons that will be clear later, the data have been arranged for increasing values of the ratio $B_D/B_T = \Gamma/E_R$.

B's (B_1, B_T, B_D): the linear approximation is valid insofar as only terms linear in the B's can be safely retained. We shall distinguish between photon absorptions that decrease the atom's kinetic energy and photon absorptions that increase it; we shall also find that, on the average, the spontaneous emission of a photon increases the atom's kinetic energy by an amount equal to the recoil energy E_R (in the linear approximation). The balance between these competing processes (cooling and heating) characterizes the steady state condition in laser cooling.

In section 6, the formulas of section 5 are applied to laser cooling, in the limit of low laser intensity, i.e. in the limit in which the stimulated emission is negligible. It is assumed that the laser photons are red detuned. The average variation of the kinetic energy $< \Delta E_K >$ of an atom, due to the absorption of a laser photon and the subsequent emission of a fluorescence one, is calculated by taking into account the different probability that the absorbed photon belongs to one of the two counter–propagating laser beams. The condition $< \Delta E_K >= 0$ yields a unique value of the atom's velocity parameter B_1 which depends on the detuning parameter δ^*, that will be defined below. This value of B_1 can be minimized as a function of the detuning parameter δ^*, thus obtaining the lowest value B_{1min}. The absorbed laser photons decrease the kinetic energy of the atoms with velocity parameter larger than B_1 (or B_{1min}) and increase the kinetic energy of the atoms with velocity parameter smaller than B_1 (B_{1min}). By identifying cB_1 with v_{rms} (or cB_{1min} with v_{rms}), we calculate the average kinetic energy $< E_K >= (1/2)Mc^2B_1^2$ in a steady state condition, or the lowest attainable average kinetic energy $< E_K >_{min}= (1/2)Mc^2B_{1min}^2$.

In A, the absorption of a counter–propagation photon by an atom is dealt within Newtonian mechanics in order to highlights the basic conceptual differences with the relativistic treatment, in spite of the fact that the two approximated calculations yields the same equations. Finally, in B, it is shown how the exact relativistic formulas can be used for describing the laser cooling of ions in storage rings.

3 Basic formulas

In [10] Schrödinger's has dealt only with the emission of photons and, in particular, he never introduced explicitly the energy difference ΔE between the two levels of the atomic transition. Therefore, in the following, the original treatment by Schrödinger is extended to the absorption case (as in [12, 13]) and the form of the equations is adapted to the problem under study. Schrödinger's approach, besides the conservations laws and special relativity, assumes that the absorption/emission process is instantaneous: more physically, that the duration of the absorption/emission process is much smaller than the lifetimes of the atomic energy levels.

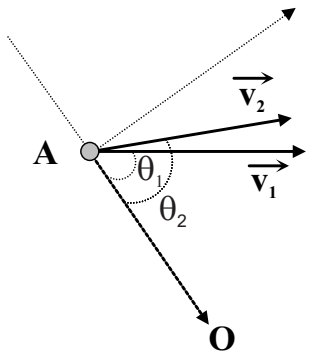

Figure 1: Emission of a photon by the atom A in motion. The photon is emitted along the direction $A \to O$. The subscript 1 denotes the quantities before the emission; the subscript 2 the quantities after the emission.

For the emission of a photon by an atom, the conservation laws are (see fig. 1):

$$E_{ph} = \gamma_1 E_1 - \gamma_2 E_2 \tag{3}$$

for energy, and

$$\gamma_1 \frac{E_1}{c^2} v_1 \cos\theta_1 = \gamma_2 \frac{E_2}{c^2} v_2 \cos\theta_2 + \frac{E_{ph}}{c} \tag{4}$$

$$\gamma_1 \frac{E_1}{c^2} v_1 \sin\theta_1 = \gamma_2 \frac{E_2}{c^2} v_2 \sin\theta_2 \tag{5}$$

for linear momentum. E_{ph} is the energy of the emitted photon; E_1 and E_2 are the rest energies of the atom before and after the emission; γ_1, γ_2 are the relativistic factors before and after the emission; v_1 and v_2 the atom's velocities before and after the emission; θ_1 and θ_2 the angles between $\vec{v}_1$ and $\vec{v}_2$ and the direction of the emitted photon. Notice that $E_1 - E_2 = \Delta E$, where ΔE is the energy difference between the two levels of the atomic transition: ΔE is a relativistic invariant, since it is given by the difference of two rest energies. After some calculations, we get:

$$E_{ph} = \frac{1}{2}\left(\frac{E_1}{\varphi_1} - \frac{E_2}{\varphi_2}\right) \tag{6}$$

where:

$$\varphi_i = \gamma_i \left(1 - \frac{v_i}{c}\cos\theta_i\right) = \gamma_i \left(1 - B_i \cos\theta_i\right); \qquad i = 1, 2 \tag{7}$$

Equation (6), by taking into account that $E_1\varphi_1 = E_2\varphi_2$, can be written as:

$$E_{ph} = \frac{1}{2}\frac{E_1^2 - E_2^2}{E_1\varphi_1} = \frac{1}{2}\frac{E_1^2 - E_2^2}{E_2\varphi_2} \tag{8}$$

Notice that $E_2 = Mc^2$ and $E_1 = Mc^2 + \Delta E$ with $\Delta E \ll Mc^2$. From now on, we shall use the dimensionless parameter $B_T = \Delta E/(2Mc^2)$: as we shall see, B_T is a threshold parameter and the velocity $v_T = cB_T$ a threshold velocity.

It is easy to verify that :

$$\frac{E_1^2 - E_2^2}{2E_2} = \frac{E_1^2 - E_2^2}{2Mc^2} = \Delta E\left(1 + \frac{\Delta E}{2Mc^2}\right) = \Delta E(1 + B_T) \tag{9}$$

$$\frac{E_1^2 - E_2^2}{2E_1} = \Delta E\left(1 - \frac{\Delta E}{2E_1}\right) \approx \Delta E(1 - B_T) \tag{10}$$

$\Delta E/(2E_1)$ differs from B_T by a term of the order of B_T^2; therefore, we shall replace in (10) the $\approx$ sign with that of the equality. Therefore, from equation

(8), we get [1]:

$$E_{ph} = \Delta E\,(1 - B_T)\,\frac{\sqrt{1 - B_1^2}}{1 - B_1 \cos\theta_1} = \Delta E\,(1 + B_T)\,\frac{\sqrt{1 - B_2^2}}{1 - B_2 \cos\theta_2} \tag{11}$$

or, in compact form:

$$E_{ph} = \frac{\Delta E}{\varphi_1}(1 - B_T) = \frac{\Delta E}{\varphi_2}(1 + B_T) \tag{12}$$

The case of absorption can be treated in the same way, starting from adequately re-written conservation equations. It turns out that the energy of the absorbed photon is given by:

$$E_{ph} = \Delta E\,(1 + B_T)\,\frac{\sqrt{1 - B_1^2}}{1 - B_1 \cos\theta_1} = \Delta E\,(1 - B_T)\,\frac{\sqrt{1 - B_2^2}}{1 - B_2 \cos\theta_2} \tag{13}$$

or by the compact equation:

$$E_{ph} = \frac{\Delta E}{\varphi_1}(1 + B_T) = \frac{\Delta E}{\varphi_2}(1 - B_T) \tag{14}$$

Both equations (11) yield the energy of the emitted photon, one in terms of the atom's velocity parameters before the emission (B_1, θ_1), the other in terms of the atom's velocity parameters after the emission (B_2, θ_2). This last equation, on the other hand, yields the energy of a photon absorbed by an atom with initial velocity parameters (B_2, θ_2) (first equation of (13) in which the subscript 1 is replaced by 2). Therefore: if an excited atom with velocity parameters (B_1, θ_1) emits a photon, the same atom, after the emission and, therefore, with velocity parameters (B_2, θ_2), can absorb a photon of the same energy.

[1] From the conceptual and pedagogical point of view, it is worth noticing that the same formula (11) can be derived in a different way. Initially, the atom is considered at rest and, by applying the conservation laws, it is found that the energy of the emitted photon is given by $E_{ph} = \Delta E\,(1 - B_T)$. When the atom is in motion with velocity $v_1 = cB_1$ along the positive direction of the x axis of the laboratory reference frame O, we consider the inertial reference frame O' comoving with the atom before emission, the axis of O' being chosen parallel and oriented in the same way as those of the laboratory reference frame O. Then, considering the energy–momentum four vector $(E_{ph}, \vec{p})$ with $p = E_{ph}/c$ of the photon, the transformation equation for E_{ph} is found. Finally, combining the two results, equation (11) is obtained. Of course, a similar calculation can be carried out for the absorption of a photon.

The variation of the atom's kinetic energy due to the emission of a photon is given by:

$$\Delta E_K^{emi} = (\gamma_2 E_2 - E_2) - (\gamma_1 E_1 - E_1) = \Delta E - E_{ph} \tag{15}$$

When the atom is at rest before emission, $\Delta E_K = \Delta E^2/2E_1 \approx \Delta E^2/2Mc^2$, since $\Delta E \ll Mc^2$. By definition, $\Delta E^2/2Mc^2$ is the recoil energy E_R. The equation:

$$E_R = \frac{1}{2} M v_R^2 \tag{16}$$

defines the recoil velocity $v_R = \Delta E/Mc = 2v_T$. Similarly, the variation of the atom's kinetic energy due to the absorption of a photon is given by:

$$\Delta E_K^{abs} = (\gamma_2 E_2 - E_2) - (\gamma_1 E_1 - E_1) = E_{ph} - \Delta E \tag{17}$$

The recoil energy in the case of absorption is the same as that in the case of emission (in the limit $\Delta E \ll Mc^2$).

4 The natural linewidth: implications

Each atomic transition has a natural linewidth Γ defined as the Full Width at Half Maximum of a Lorentzian function centered at the value of the transition energy ΔE corresponding to the highest transition probability: we shall denote this value as ΔE_M. The existence of a natural linewidth suggests to define another dimensionless parameter: $B_D = \Gamma/\Delta E_M$; B_D defines the Doppler limit in laser cooling since $B_D \Delta E_M$ describes the first order Doppler effect.

In laser cooling, we are particularly interested in the absorption of photons belonging to two counter–propagating laser beams. An atom with velocity parameter B_1 can absorb a counter–propagating photon if its transition energy ΔE^A satisfies the first of equations (13) (the superscript A stays for 'anti–parallel'). The same atom can absorb a co–propagating photon if its transition energy ΔE^P satisfies the same equation (the superscript P stays for 'parallel'). Then:

$$\Delta E^P = \Delta E^A \frac{1 - B_1}{1 + B_1} \tag{18}$$

ΔE^P is always smaller than ΔE^A for $B_1 \neq 0$ and equal to ΔE^A for $B_1 = 0$.

5 The linear approximation

In the linear approximation, the energy of the emitted photon is obtained from equations (11) with the approximation $\sqrt{1-B_1^2} \approx 1 - B_1^2/2$ and by keeping only terms linear in the B's (B_T, B_D, B_1):

$$E_{ph} = \Delta E(1 - B_T + B_1 \cos\theta_1) = \Delta E(1 + B_T + B_2 \cos\theta_2) \tag{19}$$

Similarly, the energy of the absorbed photon, in the linear approximation, is obtained from equations (13):

$$E_{ph} = \Delta E(1 + B_T + B_1 \cos\theta_1) = \Delta E(1 - B_T + B_2 \cos\theta_2) \tag{20}$$

In the linear approximation, the energy of the emitted/absorbed photon depends only on the atom's velocity component along the direction of propagation of the emitted/absorbed photon.

5.1 Variation of the atom's kinetic energy: absorption

As we have seen in equation (17), the variation of the kinetic energy of the atom due to the absorption of a photon is given by:

$$\Delta E_K = E_{ph} - \Delta E \tag{21}$$

Let us suppose that a photon with energy E_{ph} is propagating along the negative direction of the x axis. This photon can be absorbed by any atom whose velocity component $v_x = -v_1 \cos\theta_1$ satisfies the first of equations (20): the variation of the atom's kinetic energy is the same for all these atoms. This means that, in the calculations, we can consider only the cases $\theta_1 = \pi$ and $\theta_1 = 0$. Then:

$$\Delta E_K = \Delta E(B_T \pm B_1) \tag{22}$$

where the minus sign corresponds to $\theta_1 = \pi$ and the plus sign to $\theta_1 = 0$.

We define a red detuned photon as a photon with energy $E_{ph} = \Delta E_M(1 + B_T)(1 - \delta^*)$ where $\Delta E_M(1 + B_T)$ is the maximum energy of an absorbed photon when the atom is at rest before absorption and δ^* is of the same order of magnitude of or smaller than B_D (or B_T; whichever is larger): then, $E_{ph} \approx \Delta E_M(1 + B_T - \delta^*)$. Therefore, $\Delta E_M \delta^*$ is the difference between the maximum energy of a photon absorbed by the atom at rest and the energy

of the red detuned photon [2]. In the linear approximation, this photon can be absorbed in a head on collision if (first equation of (20)):

$$\Delta E^A = \Delta E_M(1 - \delta^* + B_1) \tag{23}$$

The variation of the atom's kinetic energy is given by:

$$\Delta E_K^A = \Delta E_M(B_T - B_1) \tag{24}$$

If $B_1 > B_T$, the absorption decreases the atom kinetic energy: in laser cooling, this is the cooling mechanism. If $B_1 < B_T$, the absorption increases the atom kinetic energy; in laser cooling, this is one of the heating processes at work. B_T is a threshold parameter. The threshold velocity is:

$$v_T = \frac{\Delta E_M}{2Mc} = \frac{v_R}{2} \tag{25}$$

The same atom with the same velocity parameter can absorb a co–propagating red detuned photon of the same energy. In this case, the transition energy of the atom must satisfy the equation:

$$\Delta E^P = \Delta E_M(1 - \delta^* - B_1) \tag{26}$$

and the variation of the atom's kinetic energy due to the absorption is:

$$\Delta E_K^P = \Delta E_M(B_T + B_1) \tag{27}$$

When the atom is flying in the same direction of the photon, the variation of the kinetic energy due to the absorption of a photon is always positive. In laser cooling, this is a second heating process.

5.2 Variation of the atom's kinetic energy: emission

In the case of emission the energy conservation implies that:

$$\Delta E_K = \Delta E - E_{ph} \tag{28}$$

[2] In literature, the detuning parameter δ is defined as $\delta = (\omega_l - \omega_a)$, where ω_l and ω_a are the laser and the atomic transition angular frequencies, respectively. Therefore, $\delta^* = |\delta|/\Delta E_M$.

According to equation (19), the energy of the emitted photon, in the linear approximation, is given by:

$$E_{ph} = \Delta E(1 - B_T + B_1 \cos\theta_1) \tag{29}$$

Correspondingly, the variation of the atom's kinetic energy is:

$$\Delta E_K = \Delta E(B_T - B_1 \cos\theta_1) \tag{30}$$

We see that, also for emission, B_T operates as a threshold parameter.

Given an atom with a velocity parameter B_1, it is useful to consider the average energy of the emitted photon under the hypothesis that any direction of emission is equally probable. From (19) we get:

$$\begin{aligned} < E_{ph} > &= \Delta E(1 - B_T) + \Delta E B_1 \frac{1}{4\pi} \int_0^\pi \cos\theta_1 (2\pi \sin\theta_1 d\theta_1) \\ &= \Delta E(1 - B_T) + \frac{B_1 \Delta E}{4} \left[\sin^2\theta_1\right]_0^\pi = \Delta E(1 - B_T) \end{aligned} \tag{31}$$

i.e., in the linear approximation, the average energy of the emitted photon is equal to that of the photon emitted when the emitting atom is at rest before emission.

The average variation of the atom's kinetic energy due to the emission of a photon with the same probability along any direction is given by:

$$< \Delta E_K >= \Delta E - < E_{ph} >= B_T \Delta E \approx B_T \Delta E_M = E_R \tag{32}$$

i.e., the average variation of the atom's kinetic energy due to the emission of a photon along an arbitrary direction is positive, independent from B_1, and equal to the recoil energy (in the linear approximation). This is a third heating mechanism in laser cooling.

We shall now consider a cycle composed by the absorption of a photon followed by a spontaneous emission along an arbitrary direction, taking into account the complications due to the linewidth. If an atom absorbs a photon in a head on collision, its transition energy is ΔE^A and the variation of its kinetic energy is $\Delta E_M(B_T - B_1)$ (equation (24)). On the other hand, the average variation of its kinetic energy due to the emission of a fluorescence photon is simply E_R. Then, the average overall variation of its kinetic energy due to the cycle considered is given by:

$$< \Delta E_K^A >= \Delta E_M(B_T - B_1) + E_R = \Delta E_M(2B_T - B_1) \tag{33}$$

This variation is negative for $B_1 > 2B_T$, null for $B_1 = 2B_T$ and positive for $B_1 < 2B_T$, where B_1 is the velocity parameter of the atom before absorption.

Similarly, if an atom flying in the same direction of the photon absorbs it and subsequently undergoes a spontaneous emission, the average variation of its kinetic energy is given by (equations 27, 32):

$$< \Delta E_K^P >= \Delta E_M(2B_T + B_1) \tag{34}$$

These two last equations contain all the information necessary for a quantitative treatment of laser cooling: taken together, along with the different transition probabilities for ΔE^A and ΔE^P, they describe all the cooling and heating processes, under the conditions specified in next section.

6 Laser cooling of two levels atoms

Before proceeding, it is worth recalling what are the cooling and heating mechanisms at work. If we assume that the laser photons are propagating along the negative direction of the x axis, then:

- if the atom's velocity component $v_x > v_T = v_R/2$, the absorption of a photon decreases the atom's kinetic energy (cooling mechanism);
- if the atom's velocity component $v_x < v_T = v_R/2$, the absorption of a photon increases the atom's kinetic energy (heating mechanism);
- the emission of a fluorescence photon increases, on the average, the atom's kinetic energy by an amount equal to the recoil energy E_R (heating mechanism).

We shall assume that: the laser photons are mono–energetic; the laser intensity is low enough so that stimulated emission is negligible; only the first fluorescence cycle is relevant (this means that the probability that a photon emitted by an atom is absorbed by another atom is negligible).

We suppose that the sample of atoms is illuminated by two opposite laser beams of red detuned photons for each direction axis (fig 2). For symmetry reasons, we can deal only with the two beams propagating, say, along the x direction.
As stated before, the energy of the red detuned photon is written as:

$$E_{ph} = \Delta E_M(1 + B_T)(1 - \delta^*) \approx \Delta E_M(1 + B_T - \delta^*) \tag{35}$$

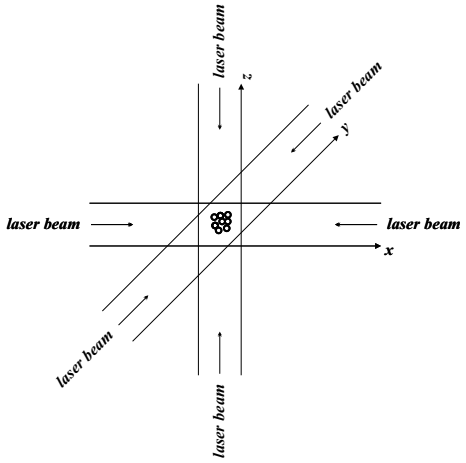

Figure 2: Schematic diagram of a 3D laser cooling setup. The atoms' sample (small circles) is illuminated by two opposite laser beams of red detuned photons for each direction axis.

If an atom with velocity parameter B_1 absorbs a laser photon, this photon belongs to one of the two opposite beams. If the photon belongs to the beam flying against the atom, the transition energy ΔE^A satisfies equation (23) (in the linear approximation):

$$\Delta E^A = \Delta E_M(1-\delta^*) + B_1 \Delta E_M \tag{36}$$

The smallest possible value of ΔE^A is $\Delta E^A = \Delta E_M(1-\delta^*)$, corresponding to $B_1 = 0$. Instead, if the photon belongs to the beam flying in the same direction of the atom, the transition energy ΔE^P satisfies the equation:

$$\Delta E^P = \Delta E_M(1-\delta^*) - B_1 \Delta E_M = \Delta E^A(1-2B_1) \tag{37}$$

The maximum possible value of ΔE^P is $\Delta E^P = \Delta E_M(1-\delta^*)$, corresponding to $B_1 = 0$ and equal to the minimum value of ΔE^A.

The transition probabilities P^A and P^P for ΔE^A and ΔE^P are different, and given by the corresponding values of the normalized Lorentzian function describing the natural line. The average variation of the kinetic energy of the atom due to the absorption of a photon and the subsequent emission of a fluorescence one, weighed by the relative probability, is calculated by using the basic equations (33, 34) that we reproduce here for convenience:

$$< \Delta E_K^A >= \Delta E_M(2B_T - B_1)P^A \tag{38}$$

for a photon flying against the atom, and

$$< \Delta E_K^P >= \Delta E_M (2B_T + B_1) P^P \tag{39}$$

for a photon flying in the same direction of the atom. Therefore, the average variation of the kinetic energy of the atom is obtained by summing the two equations (38, 39) member by member:

$$< \Delta E_K >= \Delta E_M [2B_T (P^A + P^P) - B_1 (P^A - P^P)] \tag{40}$$

Putting:

$$Q(\delta^*) = 2B_T \frac{P^A + P^P}{P^A - P^P} \tag{41}$$

$< \Delta E_K >$ will be negative if $B_1 > Q(\delta^*)$, null if $B_1 = Q(\delta^*)$ and positive if $B_1 < Q(\delta^*)$. The condition $B_1 = Q(\delta^*)$ yields:

$$B_1 = 2B_T \frac{P^A + P^P}{P^A - P^P} \tag{42}$$

i.e.

$$B_1 = 2B_T \frac{L[\Delta E_M (1 - \delta^* + B_1)] + L[\Delta E_M (1 - \delta^* - B_1)]}{L[\Delta E_M (1 - \delta^* + B_1)] - L[\Delta E_M (1 - \delta^* - B_1)]} \tag{43}$$

where the L is the normalized Lorentzian function describing the natural line shape. After some manipulations we get:

$$B_1^2 = \frac{B_T (B_D^2 + 4\delta^{*^2})}{4(\delta^* - B_T)} \tag{44}$$

with $\delta^* > B_T$. This condition means that the energy of the red detuned photon $E_{ph} = \Delta E_M (1 + B_T - \delta^*)$ must be smaller than ΔE_M. Then:

$$B_1 = \frac{1}{2} \sqrt{B_T \frac{B_D^2 + 4\delta^{*^2}}{\delta^* - B_T}} \tag{45}$$

The laser beams will reduce the average kinetic energy of the atoms with velocity parameter larger than B_1 and will increase the average kinetic energy of the atoms with velocity parameter smaller than B_1. The smallest value B_{1min} of B_1 is obtained for

$$\delta^* = B_T + \frac{1}{2} \sqrt{4B_T^2 + B_D^2} \tag{46}$$

and is given by:

$$B_{1min} = B_T \sqrt{2 + 2\sqrt{1 + \frac{B_D^2}{4B_T^2}}} \tag{47}$$

These results are valid for any value of the ratio $B_D/B_T = \Gamma/E_R$. It is interesting to consider three cases: $B_D \gg B_T$, $B_D = B_T$ and $B_D \ll B_T$. In the first case, B_{1mim} is obtained for $\delta^* \approx B_D/2$; in the second, for $\delta^* = 2.12\, B_T$; in the third for $\delta^* \approx 2B_T$.

The absorption of a laser photon depends only on the atom's velocity component v_x. Since the atoms' sample is under the action of two counter–propagating laser beams along the x axis, the distribution of v_x is symmetric around $v_x = 0$: therefore, the atoms' sample in the steady state can be described as if all the atoms have velocity component $v_x = \pm v_{x_{rms}}$. When this velocity is equal to $\pm cB_1$, the average kinetic energy of the atoms can not be reduced further. Then, by putting $cB_1 = v_{x_{rms}}$, we get:

$$< E_K >= \frac{1}{2} M < v_x^2 >= \frac{1}{2} M c^2 B_1^2 = \frac{1}{16} E_R \frac{B_D^2 + 4\delta^{*2}}{B_T(\delta^* - B_T)} \tag{48}$$

The lowest attainable kinetic energy will be:

$$< E_K >_{min} = \frac{1}{2} M c^2 B_{1min}^2 = \frac{1}{2} E_R \left(1 + \sqrt{1 + \frac{B_D^2}{4B_T^2}}\right) \tag{49}$$

This equation implies that $< E_K >_{min} \rightarrow E_R$ as $B_D/B_T \rightarrow 0$ (Fig. 3) [3].
When $\Gamma = 0$, red detuned photons can be absorbed only by atoms flying against the photons and the minimum condition $< E_K >_{min} = E_R$ can be derived directly from equation (33). In fact, according to (33), the average variation of the atom's kinetic energy due to the absorption of a photon and the subsequent emission of a fluorescence one is zero for $B_1 = 2B_T$. Then, the average kinetic energy of an atom in the steady state condition in given by:

$$< E_K >= \frac{1}{2} M c^2 B_1^2 = \frac{1}{2} M c^2 (2B_T)^2 = E_R \tag{50}$$

In three dimensions, the minimum attainable average kinetic energy is three times that in one dimension.

[3] It should be reminded that a small natural linewidth corresponds to long lifetimes of the excited state. In these cases, the time necessary to cool the sample becomes very long: usually, an auxiliary transition is used in the early stages of cooling.

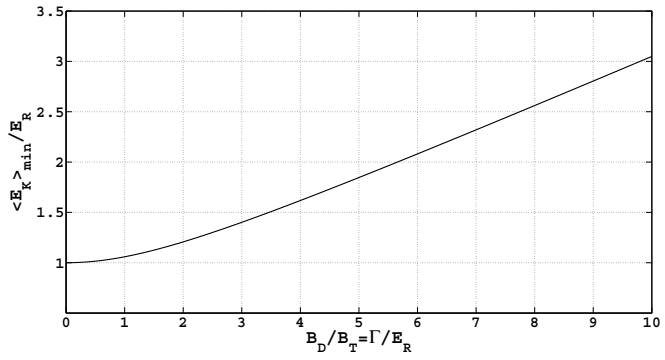

Figure 3: The lowest average kinetic energy goes to E_R as Γ/E_R goes to zero.

6.1 Comparison with the semiclassical theory

From equations (48, 49), we get, dividing member by member:

$$\frac{< E_K >}{< E_K >_{min}} = \frac{1}{8} \frac{B_D^2 + 4\delta^{*2}}{(\delta^* - B_T)(B_T + \frac{1}{2}\sqrt{4B_T^2 + B_D^2})} \tag{51}$$

If $B_T = 0$, this equation reduces to:

$$\frac{< E_K >}{< E_K >_{min}} = \frac{1}{2}\left(\frac{B_D}{2\delta^*} + \frac{2\delta^*}{B_D}\right) \tag{52}$$

In literature, the detuning parameter δ is defined as $\delta = \omega_l - \omega_a$ where ω_l and ω_a are the laser and the atomic transition angular frequencies. Therefore: $\delta^* = \hbar|\delta|/\Delta E_M$. Taking into account this relation and that $\Gamma = \hbar\gamma$ (where γ is the natural width expressed in terms of angular frequency), equation (52) assumes the form:

$$\frac{< E_K >}{< E_K >_{min}} = \frac{1}{2}\left(\frac{\gamma}{2|\delta|} + \frac{2|\delta|}{\gamma}\right) \tag{53}$$

This is the result of the semiclassical theory of two levels atoms (at low laser intensities): it is a limiting case of the present treatment based on conservation laws. Furthermore, from equation (49), we get, for $B_D \gg B_T$:

$$< E_K >_{min} = \frac{\hbar\gamma}{4} \tag{54}$$

which is the lowest attainable average kinetic energy predicted by the semiclassical theory. Finally, inserting this value in equation (53):

$$< E_k >= \frac{\hbar\gamma}{8}\left(\frac{\gamma}{2|\delta|} + \frac{2|\delta|}{\gamma}\right) \tag{55}$$

which yields the average kinetic energy of the atom according to the semiclassical theory.

The comparison with the semiclassical theory can be visualized by three figures. Fig. 4 shows that the two treatments are practically indistinguishable when $B_D \gg B_T$.

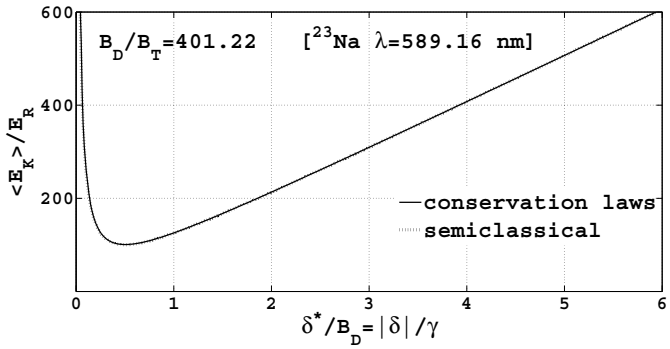

Figure 4: Comparison between the semiclassical theory (dotted line) and the present treatment for the case of the 589.16 nm transition of ^{23}Na: the ratio between the average kinetic energy and the recoil energy is plotted as a function of the detuning parameter.

In the semiclassical theory, it is explicitly assumed that the photon momentum is negligible with respect to the atomic one. This approximation implies that the kinetic energy of the atom is much larger than the recoil energy. Consequently, in the semiclassical theory both the average kinetic energy and the lowest attainable average kinetic energy depend only on γ (on B_D in the language of this paper) equations (55) and (54). Instead, in the present treatment, they depend on both B_D and B_T (equations (48, 49)). However, when $B_D \approx B_T$ the differences between the two treatments are evident (fig. 5).
Finally, the lowest attainable kinetic energy is systematically lower in the semiclassical theory (Fig. 6): the ratio between the value predicted by the

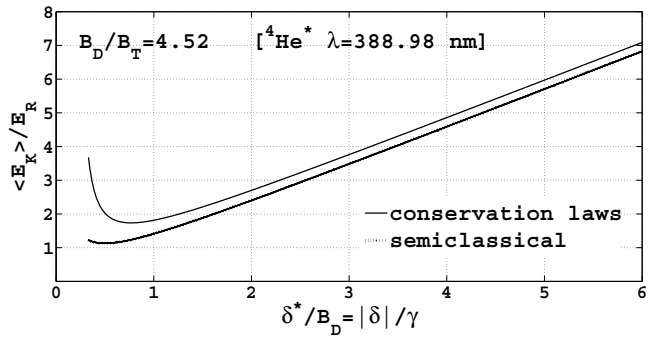

Figure 5: Comparison between the semiclassical theory (dotted line) and the present treatment for the case of the 388.98 nm transition of $^4He^*$: the ratio between the average kinetic energy and the recoil energy is plotted as a function of the detuning parameter. The * remind us that the starting level of the transition is a metastable one.

semiclassical theory and the one obtained by conservation laws tends to one for large values of the ratio B_D/B_T but drops dramatically as $B_D/B_T \to 0$. Of course, this behavior is due to the fact that B_T does not enter into the semiclassical theory.

7 Comparison with quantum mechanical treatments

Quantum mechanical treatments of laser cooling of two levels atoms at low laser intensities have been carried out by several authors. Wineland and Itano [3] deal with both free and bound atoms and begin with formulas based on conservation laws applied to the absorption/emission of photons by atoms. The difference with the present treatment lies in the fact that Wineland and Itano's calculation of the steady state average kinetic energy requires the knowledge of the atoms' velocity distribution, assumed to be always Gaussian. For free atoms, when the linewidth is much smaller than the recoil energy, they found that it should be possible to achieve average kinetic energies lower than the recoil energy. However, Wineland and Itano stress that it is difficult to obtain "the proper conditions under which these results hold [3, p. 1525]." Instead, in the present paper, the calculation of

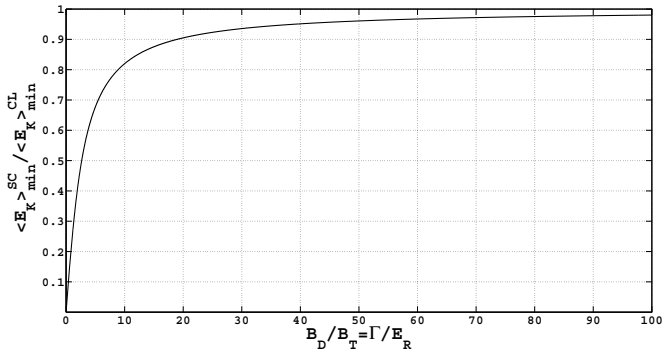

Figure 6: Ratio between the lowest values of the average kinetic energy predicted by the semiclassical theory $< E_K >^{SC}_{min}$ and the conservation laws $< E_K >^{CL}_{min}$ plotted as a function of the ratio between the natural linewidth Γ and the recoil energy E_R.

the average kinetic energy is based on the position $cB_1 = v_{x_{rms}}$ where cB_1 is the atom's velocity which zeroes the average kinetic energy variation due to an absorption–emission cycle: no knowledge of the velocity distribution is required. As a consequence, as shown above, when $\Gamma/E_R \to 0 < E_K > \to E_R$: E_R is the lowest kinetic energy allowed by conservation principles in laser cooling of two levels atoms at low laser intensities. In the second part of their paper, Wineland and Itano show that the quantum mechanical treatment of free atoms does not alter the picture given in the first part of their paper.

The quantum mechanical treatment by Castin, Wallis and Dalibard [4] does not make any assumption on the atoms' velocity distribution and is valid for any value of the ratio between the natural linewidth and the recoil energy. As in the present paper, they show that the semiclassical theory is valid as long as the natural linewidth is much larger than the recoil energy; however, differently from the present paper, they found that, for very narrow transition lines, the lowest attainable average kinetic energy is about $0.5E_R$ instead of E_R. It is not clear to me why a quantum mechanical treatment yields this limit value which seems to be incompatible with the conservation laws.

8 Conclusions

The straightforward application of energy and linear momentum conservation to the absorption/emission of photons by atoms allows to find out the essential features of laser cooling of two levels atoms at low laser intensities. The lowest attainable average kinetic energy of the atoms depends on the ratio Γ/E_R between the natural linewidth and the recoil energy and tends to E_R as Γ/E_R tends to zero (in one dimension). This treatment, like the quantum mechanical ones, is valid for any value of the ratio between the natural linewidth and the recoil energy and contains the results of the standard semiclassical theory of laser cooling as the limiting case in which the recoil energy is negligible with respect to the natural linewidth.

A Absorption of a photon in Newtonian mechanics

Suppose that an atom, flying in the positive direction of the x axis, absorbs a counter–propagating photon with energy E_{ph}. The conservation of linear momentum reads:

$$Mv_1 - \frac{E_{ph}}{c} = Mv_2 \tag{56}$$

where v_1 and v_2 are the component of the atom velocity along the x axis, before and after the absorption, respectively. Notice how the light speed c, extraneous to classical mechanics, enters this equation through the photon linear momentum. The conservation of energy reads:

$$\frac{1}{2}Mv_1^2 + E_{ph} = \frac{1}{2}Mv_2^2 + \Delta E \tag{57}$$

where ΔE is the energy difference between the two levels of the atomic transition. From (56):

$$v_2 = v_1 - \frac{E_{ph}}{Mc} \tag{58}$$

Substituting this value in (57), we get:

$$E_{ph} = \Delta E + \frac{1}{2Mc^2}E_{ph}^2 - B_1 E_{ph} \tag{59}$$

where $B_1 = v_1/c$. If we write:

$$E_{ph} = \Delta E(1+\alpha) \tag{60}$$

equation (59) becomes:

$$\Delta E(1+\alpha) = \Delta E + \frac{\Delta E}{2Mc^2}\Delta E(1+\alpha)^2 - B_1\Delta E(1+\alpha) \tag{61}$$

Putting $B_T = \Delta E/2Mc^2$ and retaining only terms of the first degree in α, B_T, B_1 (linear approximation) we obtain:

$$\alpha = B_T - B_1 \tag{62}$$

Therefore:

$$E_{ph} = \Delta E(1 + B_T - B_1) \tag{63}$$

which is identical to the first of equations (20) in the case of $\theta_1 = \pi$. From (57) and (63), the variation of the atom's kinetic energy is obtained:

$$\Delta E_K = E_{ph} - \Delta E = \Delta E(B_T - B_1) \tag{64}$$

If $B_1 = 0$, $E_{ph} = \Delta E(1 + B_T)$ and $\Delta E_K = \Delta E^2/2Mc^2 = E_R$. E_R is the recoil energy.

The rest energy of special relativity Mc^2 appears in these equations as a consequence of having considered the absorption of a photon. Of course, in classical mechanics, the physical meaning of Mc^2 remains obscure.

The energy of an emitted can be derived in a similar way, by writing down the adequately re–written conservation equations.

B Ions in storage rings

As recalled in the Introduction, ions at relativistic speeds in storage rings can be 'cooled', i.e. their kinetic energy decreased, by absorption of laser photons. Here, we shall develop a calculation just to show how the 'basic formulas' of section 3 can be applied to this relativistic case.

If an ion with velocity parameter $B_1 = v_1/c$ absorbs a photon of a counter–propagating laser beam in a head–on collision, the energy of the absorbed photon is given by (equation (13) with $\theta_1 = \pi$):

$$E_{ph}^{abs} = \Delta E^A(1 + B_T)\sqrt{\frac{1 - B_1}{1 + B_1}} = \Delta E^A(1 + B_T)k_1 \approx k_1\Delta E^A \tag{65}$$

if $k_1 = \sqrt{\frac{1-B_1}{1+B_1}} < 1$ is of the order of unity (as we shall see) and B_T of the order of 10^{-10}. On the other hand, we have:

$$E_{ph}^{abs} = k_1 \Delta E^A = E_{ph}^{laser} \tag{66}$$

where E_{ph}^{laser} is the energy of the laser photon. The variation of the ion's kinetic energy due to the absorption of a photon is given by (equation 17):

$$\Delta E_K^{abs} = E_{ph}^{abs} - \Delta E^A = -\Delta E^A (1 - k_1) = -E_{ph}^{laser} \frac{1 - k_1}{k_1} \tag{67}$$

After the absorption, the atom's velocity parameter B_2 obeys the equation (see equation 13):

$$(1 + B_T)\sqrt{\frac{1 - B_1}{1 + B_1}} = (1 - B_T)\sqrt{\frac{1 - B_2}{1 + B_2}} \tag{68}$$

which yields:

$$B_2 = \frac{B_1 - 2B_T + B_1 B_T^2}{1 - 2B_1 B_T + B_T^2} \approx \frac{B_1 - 2B_T}{1 - 2B_1 B_T} \tag{69}$$

or:

$$v_2 \approx \frac{v_1 - v_R}{1 - v_1 v_R / c^2} \tag{70}$$

where v_R is the recoil velocity. Let us now consider an inertial frame O' comoving with the atom before absorption, the axis of O' being chosen parallel and oriented in the same way as those of the laboratory reference frame O: O' is in motion with respect to O with the atom's velocity $v_1 = B_1 c$ along the positive direction of the common $x \equiv x'$ axis: the laser beam is propagating along the negative direction of the $x \equiv x'$ axis. In O', after the absorption, the ion is in motion with velocity $v'_{2x} = -v_R$. Then, since, $v_2 = v_{2x}$, equation (70) is the relativistic transformation for the x component of the velocity, as it must be. Equation (70) is an approximated one. This approximation reflects the fact that the value of v_R for the recoil velocity is derived by writing $E_R = (1/2)Mv_R^2$, i.e. by applying classical mechanics: rigorously, $E_R = Mc^2(\gamma_R - 1)$, where $\gamma_R = 1/\sqrt{1 - v_R^2/c^2}$.

The excited ion will emit a fluorescence photon with the same probability in any direction. The average energy of the emitted photon can be calculated as shown in section 5.2: but now, we have to use the exact relativistic

formulas. Then:

$$\begin{aligned} < E_{ph}^{emi} > &= \Delta E^A (1 - B_T)\sqrt{1 - B_2^2}\,\frac{1}{4\pi}\int_0^\pi \frac{2\pi \sin\theta}{1 - B_2\cos\theta}d\theta \\ &= \Delta E^A (1 - B_T) Y \end{aligned} \tag{71}$$

where

$$Y = \frac{\sqrt{1 - B_2^2}}{2B_2}[\ln(1 + B_2) - \ln(1 - B_2)] \tag{72}$$

is, as we shall see, of the order of unity. Therefore:

$$\begin{aligned} < E_K^{emi} > &= \Delta E_A - < E_{ph}^{emi} > = \Delta E^A - \Delta E^A (1 - B_T) Y \approx \\ &\approx \Delta E^A (1 - Y) = E_{ph}^{laser}\frac{1 - Y}{k_1} \end{aligned} \tag{73}$$

And, finally:

$$\Delta E_K^{abs} - < \Delta E_K^{emi} > = -E_{ph}^{laser}\frac{2 - k_1 - Y}{k_1} \tag{74}$$

In order to evaluate the orders of magnitude involved, we shall use the data of [14], reported in table 2.

ion	$\lambda_{transition}$ (nm)	$B_1 = v_1/c$	$B_T = E_R/\Delta E_M$	λ_{laser} (nm)
$^7Li^+$	548.5	6.4×10^{-2}	1.73×10^{-10}	584.8

Table 2: Data from [14]. The transition used is $^3S_1(F = \frac{5}{2}) \rightarrow {}^3P_2(F = \frac{7}{2})$. λ_{laser} is the wavelength of a counter–propagating dye laser beam.

In terms of photon energy, $\lambda_{transition}$ corresponds to 2.26 eV and λ_{laser} to 2.12 eV. Therefore, the laser photons are greatly red detuned: as a matter of fact, the detuning parameter δ^* is 6.19×10^{-2}, i.e. several orders of magnitude larger than those considered in laser cooling of an atomic gas, where δ^* is of the order of B_D, i.e. of 10^{-8} (see table 1). Then, $k_1 = 0.9379$ and $Y = 0.9993$. It follows that $\Delta E_K^{abs} = -1.40 \times 10^{-2}$ eV, and $< \Delta E_K^{emi} > = 1.58 \times 10^{-3}$ eV and the difference $\Delta E_K^{abs} - < \Delta E_K^{emi} > = -1.387 \times 10^{-1}$ eV. Therefore, in a cycle constituted by the absorption of a photon followed by the emission of a fluorescence one, the kinetic energy of an ion decreases, on the average, by an amount of 1.387×10^{-1} eV and ten thousand cycles will decrease the ion kinetic energy by 1.387 keV. Since the lifetime of the $^3P_2$ state is 43 ns, it is understandable how a long enough interaction time of the stored ions with

the laser beam can produce a decrease of the ions kinetic energy of several keV [14, p. 2901].

It is worth noticing that $\Delta E^A = E_{ph}^{laser}/k_1 = 2.26\,\mathrm{eV} = E_{ph}^{transition}$. In other words, $\Delta E_A = \Delta E_M(1 + B_T) \approx \Delta E_M$, where ΔE_M is the maximum transition energy described by the Lorentzian shape of the natural linewidth. As a matter of fact in [14, p. 2902], the ions velocity has been chosen in order to excite exactly the transition at $\lambda = 548.4$ nm (table 2). This point well illustrates the role played by the conservation laws: the photon is greatly red detuned, but the missing energy for the transition $\Delta E_M - E_{ph}^{laser}$ is supplied by the atom's kinetic energy which, consequently, diminishes by the same amount.

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