

On electromagnetic induction

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Abstract. A general law for electromagnetic induction phenomena is derived from Lorentz force and Maxwell equation connecting electric field and time variation of magnetic field. The derivation provides with a unified mathematical treatment the statement according to which electromagnetic induction is the product of two independent phenomena: time variation of magnetic field and effects of magnetic field on moving charges. The general law deals easily - without *ad hoc* assumptions - with typical cases usually considered as exceptions to the flux rule and contains the flux rule as a particular case.

Contents

1	Introduction	3
2	A general law for electromagnetic induction	3
2.1	A definition of <i>emf</i>	4
2.2	\vec{A} versus \vec{B}	6
3	How the law works	7
3.1	General features	7
3.2	How it works in specific cases	9
3.2.1	When a bar is moving	9
3.2.2	The Faraday disc: 1	11

3.2.3	The unipolar induction	13
3.2.4	The flux varies, but...	14
3.2.5	The ‘rocking plates’	15
3.2.6	The Corbino disc	16
3.2.7	The Faraday disc: 2	18
4	General law and flux rule	19
5	Conclusions	20

1 Introduction

It is, in general, acknowledged that the theoretical treatment of electromagnetic induction phenomena presents some problems when part of the electrical circuit is moving. Some authors speak of exceptions to the flux rule;¹ others save the flux rule by *ad hoc* choices of the integration line over which the induced *emf* is calculated. Several attempts to overcome these difficulties have been made; a comprehensive one has been performed by Scanlon, Henriksen and Allen.² However, their treatment - as others - fails to recognize that one must distinguish between the velocity of the circuit elements and the velocity of the electrical charges (see section 2 below). Therefore, these authors reestablish the flux rule and, consequently, do not solve the problems posed by its application.

Since 1992, I have been teaching electromagnetism in a course for Mathematics students and I had to deal with the problems outlined above. I have found that it is possible to get a general law for electromagnetic induction that contains the standard flux rule as a particular case.

The matter has conceptual relevance; it has also historical and epistemological aspects that deserve to be investigated. Therefore, it is, perhaps, worthwhile to submit the following considerations to the attention of a public wider than that of my students.

2 A general law for electromagnetic induction

Textbooks show a great variety of positions about how the flux rule can be applied to the known experimental phenomena of electromagnetic induction. Among the more lucid approaches, let us refer to the treatment given by Feynman, Leighton and Sands in the *Feynman Lectures on Physics*. They write:

¹ R. Feynman, R. Leighton and M. Sands *The Feynman Lectures on Physics*, vol. II, (Addison Wesley, Reading, Ma., 1964), pp. 17 - 2,3.

² P.J. Scanlon, R.N. Henriksen and J.R. Allen, "Approaches to electromagnetic induction", Am. J. Phys., 37, (1969), 698 - 708.

In general, the force per unit charge is $\vec{F}/q = \vec{E} + \vec{v} \times \vec{B}$. In moving wires there is the force from the second term. Also, there is an \vec{E} field if there is somewhere a changing magnetic field. They are two independent effects, but the emf around the loop of wire is always equal to the rate of change of magnetic flux through it.³

This sentence is followed by a paragraph entitled *Exceptions to the “flux rule”*, where the authors treat two cases - the Faraday disc and the ‘rocking plates’ - both characterized by the fact that there is a part of the circuit in which the *material of the circuit* is changing. As the authors put it, at the end of the discussion:⁴

The ‘flux rule does not work in this case. It must be applied to circuits in which the *material* of the circuit remain the same. When the material of the circuit is changing, we must return to the basic laws. The *correct* physics is always given by the two basic laws

$$\vec{F} = e(\vec{E} + \vec{v} \times \vec{B}) \quad (1)$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad (2)$$

2.1 A definition of *emf*

In order to try shedding some more light on the subject, let us begin with the acknowledgement that the expression of Lorentz force

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B}) \quad (3)$$

not only gives meaning to the fields solutions of Maxwell equations when applied to point charges, but yields new predictions.⁵

³ R. Feynman, R. Leighton and M. Sands *The Feynman Lectures on Physics*, vol. II, (Addison Wesley, Reading, Ma., 1964), p. 17 - 2.

⁴ Ibidem, p. 17 - 3.

⁵ The fact that the expression of Lorentz force can be derived by considering an inertial frame in which the charge is at rest and by *assuming* that the force acting on it is simply given by $\vec{F}' = q\vec{E}'$, does not change the matter.

The velocity appearing in the expression of Lorentz force is the velocity of the charge: from now on, we shall use the symbol \vec{v}_{charge} for distinguishing the charge velocity from the velocity \vec{v}_{line} of the circuit element that contains the charge. *This is a basic point of the present treatment.*

Let us consider the integral of $(\vec{E} + \vec{v}_{charge} \times \vec{B})$ over a closed loop:

$$\mathcal{E} = \oint_l (\vec{E} + \vec{v}_{charge} \times \vec{B}) \cdot d\vec{l} = \oint_l \vec{E} \cdot d\vec{l} + \oint_l (\vec{v}_{charge} \times \vec{B}) \cdot d\vec{l} \quad (4)$$

This integral yields the work done by the electromagnetic field on a unit positive point charge along the closed path considered. It presents itself as the *natural* definition of the electromotive force, within the Maxwell - Lorentz theory: $emf = \mathcal{E}$.

Let us now calculate the value of \mathcal{E} given by equation (4). The calculation of the first integral appearing in the third member of equation (4) yields:

$$\oint_l \vec{E} \cdot d\vec{l} = \int_S \text{rot } \vec{E} \cdot \hat{n} dS = - \int_S \frac{\partial \vec{B}}{\partial t} \cdot \hat{n} dS \quad (5)$$

where S is any surface having the line l as contour and where we have made use of Maxwell equation (2). The calculation of the last integral of equation (5) yields:

$$\oint_l \vec{E} \cdot d\vec{l} = - \frac{d}{dt} \int_S \vec{B} \cdot \hat{n} dS - \oint_l (\vec{v}_{line} \times \vec{B}) \cdot d\vec{l} \quad (6)$$

where \vec{v}_{line} is the velocity of the circuit element $d\vec{l}$.⁶ Notice that equation (6) is the result of a theorem of vectorial calculus that is valid for any vector field \vec{G} , like the magnetic field, for which $\text{div } \vec{G} = 0$. Therefore, we get:

$$\mathcal{E} = \left[- \frac{d}{dt} \int_S \vec{B} \cdot \hat{n} dS - \oint_l (\vec{v}_{line} \times \vec{B}) \cdot d\vec{l} \right] + \oint_l (\vec{v}_{charge} \times \vec{B}) \cdot d\vec{l} \quad (7)$$

This equation says that:

1. The induced *emf* is, in general, given by three terms.

⁶ A. Sommerfeld, *Lectures in Theoretical Physics*, vol. II, (Academic Press, New York) 1950, pp. 130 - 132; ibidem, vol. III, p. 286.

2. The first two - grouped under square brackets for underlining their common mathematical and physical origin - come from the line integral $\oint_l \vec{E} \cdot d\vec{l}$, whose value is controlled by Maxwell equation (2) through equation (5). Accordingly, *their sum must be zero when the magnetic field does not depend on time*. In this case, the law assumes the simple form:

$$\mathcal{E} = \oint_l (\vec{v}_{charge} \times \vec{B}) \cdot d\vec{l} \quad (8)$$

3. The third term comes from the magnetic component of Lorentz force; we shall see later how this term may be different from zero.
4. The flux rule is contained in the general law as a particular case.

The general law (7) can be written also in terms of the vector potential \vec{A} . If we put

$$\vec{E} = -grad \varphi - \frac{\partial \vec{A}}{\partial t} \quad (9)$$

in the first integral of the third member of equation (4), we get at once

$$\mathcal{E} = - \oint_l \frac{\partial \vec{A}}{\partial t} \cdot d\vec{l} + \oint_l (\vec{v}_{charge} \times \vec{B}) \cdot d\vec{l} \quad (10)$$

since $\oint_l grad \varphi \cdot d\vec{l} = 0$. When the magnetic field does not depend on time ($\partial \vec{B} / \partial t = 0$), we have:

$$\oint_l \vec{E} \cdot d\vec{l} = - \oint_l \frac{\partial \vec{A}}{\partial t} \cdot d\vec{l} = - \int_S rot \frac{\partial \vec{A}}{\partial t} \cdot \hat{n} dS = 0 \quad (11)$$

since $rot (\partial \vec{A} / \partial t) = 0$, because $\partial \vec{B} / \partial t = 0$. Therefore, the only surviving term in the expression of the induced *emf* is coming from the magnetic component of Lorentz force and the law assumes the form given by equation (8).

2.2 \vec{A} versus \vec{B}

Traditionally, textbooks use equations containing the magnetic field in dealing with induction phenomena: the *emf* in a loop is partially (general law) or totally (flux

rule) dependent on a surface integral of \vec{B} . The laws are predictive, but they are not causal laws. The values of \vec{B} over the surface of integration cannot be causally related to the value of the *emf* around the circuit because \vec{B} acts at a distance (with an infinite propagation velocity): *in this case*, \vec{B} is not a *good* field, since a *good* field, in Feynman's words, can be defined as

... a set of numbers we specify in such a way that what happens *at a point* depends only on the numbers *at that point*. We do not need to know any more about what's going on at other places.⁷

On the contrary, the same equations written in terms of the vector potential, are causal laws since they relate the value of the *emf* around the circuit to the values of $-\partial\vec{A}/\partial t$ at the points of the loop: the vector potential is, in this case, a *good* field.

3 How the law works

3.1 General features

Let us come back to the general law (7 or 10). The charge velocity appearing in these equations contributes to build up, through the factor $\vec{v}_{charge} \times \vec{B}$ the induced electromotive field. The charge velocity is given by:

$$\vec{v}_{charge} = \vec{v}_{line} + \vec{v}_{drift} \quad (12)$$

where \vec{v}_{drift} is the drift velocity.

Therefore, the general equation for electromagnetic induction assumes the form:

$$\begin{aligned} \mathcal{E} = & \left[-\frac{d}{dt} \int_S \vec{B} \cdot \hat{n} dS - \oint_l (\vec{v}_{line} \times \vec{B}) \cdot d\vec{l} \right] + \\ & + \oint_l (\vec{v}_{line} \times \vec{B}) \cdot d\vec{l} + \oint_l (\vec{v}_{drift} \times \vec{B}) \cdot d\vec{l} \end{aligned} \quad (13)$$

⁷ R. Feynman, R. Leighton and M. Sands *The Feynman Lectures on Physics*, vol. II, (Addison Wesley, Reading, Ma., 1964), p. 15 - 7. Feynman, uses the term 'real field' instead of 'good field'. Feynman deals with this problem in discussing the Bohm - Aharanov effect. It is interesting to see that an identical situation arises in classical electromagnetism.

or, in terms of the vector potential:

$$\mathcal{E} = - \oint_l \frac{\partial \vec{A}}{\partial t} \cdot d\vec{l} + \oint_l (\vec{v}_{line} \times \vec{B}) \cdot d\vec{l} + \oint_l (\vec{v}_{drift} \times \vec{B}) \cdot d\vec{l} \quad (14)$$

When the circuit is made by a loop of wire, equation (13) reduces to equation

$$\mathcal{E} = \left[-\frac{d}{dt} \int_S \vec{B} \cdot \hat{n} dS - \oint_l (\vec{v}_{line} \times \vec{B}) \cdot d\vec{l} \right] + \oint_l (\vec{v}_{charge/line} \times \vec{B}) \cdot d\vec{l} \quad (15)$$

and equation (14) to equation

$$\mathcal{E} = - \oint_l \frac{\partial \vec{A}}{\partial t} \cdot d\vec{l} + \oint_l (\vec{v}_{charge/line} \times \vec{B}) \cdot d\vec{l} \quad (16)$$

because the drift velocity is always parallel to the line element $d\vec{l}$ and, consequently, the integral containing it is zero. We have introduced the new notation $v_{charge/line}$ for remembering that we are dealing with the velocity of the charge that, in this case, can be replaced by the velocity of the circuit element that contains it. It is worth stressing again that, when the magnetic field does not depend on time, equation (15) cannot be read in terms of flux variation, since the sum of the first two terms under square brackets is zero as it is the equivalent term of equation (16) containing the time derivative of the vector potential.

When part of the circuit is made by extended material, the calculation of the drift velocity contribution to the induced *emf* is not an easy task, since the distribution of current lines in the material may be very complicated.⁸ In section 3.2.6 and 3.2.7 we shall treat particularly simple cases with a circular symmetry. In all other cases, we shall neglect the drift velocity in the calculation of the induced *emf* (look at the end of section 3.2.6 for a further discussion of this point).

⁸ The distribution of currents in extended materials have been studied since the mid of eighteenth century. The following references are those that I know: E. Jochmann, "On the electric currents induced by a magnet in a rotating conductor", *Phil. Mag.*, **27**, (1864), 506 - 528; *Phil. Mag.*, **28**, (1864), 347 - 349; H. Hertz, 'On induction in rotating spheres', in Miscellaneous papers, (J.A. Barth, Leipzig) 1895, pp. 35 - 126. A discussion of some aspects of Hertz's work can be found in: Jed Z. Buchwald, The creation of scientific effects - Heinrich Hertz and electric waves, (The University of Chicago Press, Chicago and London) 1994, pp. 95 - 103; the papers by Boltzmann and Corbino quoted in footnotes 12; 13. V. Volterra, "Sulle correnti elettriche in una lamina metallica sotto l'azione di un campo magnetico", *Il Nuovo Cimento*, **9**, 23 - 79 (1915). One should also see the literature on eddy currents.

3.2 How it works in specific cases

We shall now discuss some cases widely treated in literature, in order to see how the general law (and the flux rule) can be applied to specific problems.

3.2.1 When a bar is moving

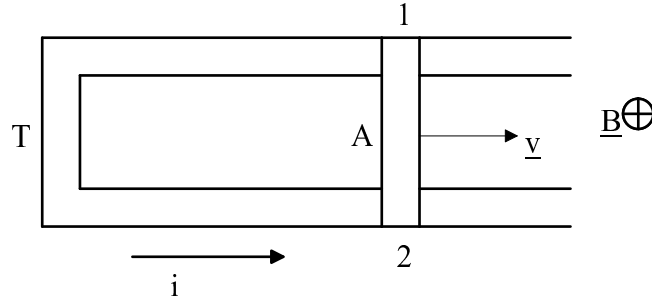


Figure 1: the conducting bar A , of length a , moves with constant velocity \vec{v} in a uniform and constant magnetic field \vec{B} perpendicular to and entering the page. The bar A slides on the conducting frame T . A steady current i circulates in the circuit $T21$.

Let us consider the circuit of fig. 1. The conducting bar A , of length a , slides with constant velocity \vec{v} over the conducting frame T in a uniform and constant magnetic field \vec{B} perpendicular to the plane of the frame and entering the page: the frame T is at rest with respect to the source of the magnetic field. A steady current flows along the direction indicated in the figure. Notice that in this case we can neglect without approximation the drift velocity of the charge because it is always directed along the integration line (owing to the Hall effect). According to the general law (16), the first integral is zero as zero is sum of the first two terms of equation (15): as a matter of fact, if we choose the counterclockwise direction for the line integral, we obtain for the sum of these two terms:

$$Bav_{line} - Bav_{line} = 0 \quad (17)$$

The induced electromotive force is then given by:

$$\mathcal{E} = Bav_{charge/line} \quad (18)$$

The surviving term is the one coming from the magnetic component of Lorentz force. Moreover, the theory predicts that the *emf* is localized into the bar: the bar acts as a battery and, as a consequence, between the two points 1 and 2 of the frame that are in contact with the bar, one should measure a potential difference given by $V_1 - V_2 = \mathcal{E} - ir$, where i is the circulating current and r is the resistance of the bar.

Finally, it is worth stressing that the energy balance shows that the magnetic component of Lorentz force plays the role of an intermediary: the electrical energy dissipated in the circuit comes from the work done by the force that must be applied to keep the bar moving with constant velocity.

Let us now recall how the flux rule deals with this case. It predicts an *emf* given by Bav_{line} . In the light of the general law (15) and of its discussion, we understand why the flux rule predicts correctly the value of the *emf*: the reason lies in the fact that the two line integrals under square brackets cancel each other. However, we have shown above that the physics embedded in the general equation (15) forbids to read equation (18) as the result of:

$$\mathcal{E} = Bav_{line} + [-Bav_{line} + Bav_{line}] = Bav_{line} \quad (19)$$

that leaves operative the first term coming from the flux variation.

Before leaving this subject, it is worth saying something about the long debate about the meaning of ‘localized *emf*’. With reference to the moving bar, we can proceed as follows. The points 1 and 2 divide the circuit into two parts: the one on the left obeys Ohm law $\Delta V = iR$ and the current enters it at point 1 at higher potential and leaves it at point 2 at lower potential; the part on the right does not obey Ohm law and the current enters it at point 2 at lower potential and leaves it at point 1 at higher potential. We say that the *emf* is localized in the right part of the circuit: it can be experimentally distinguished from the other.

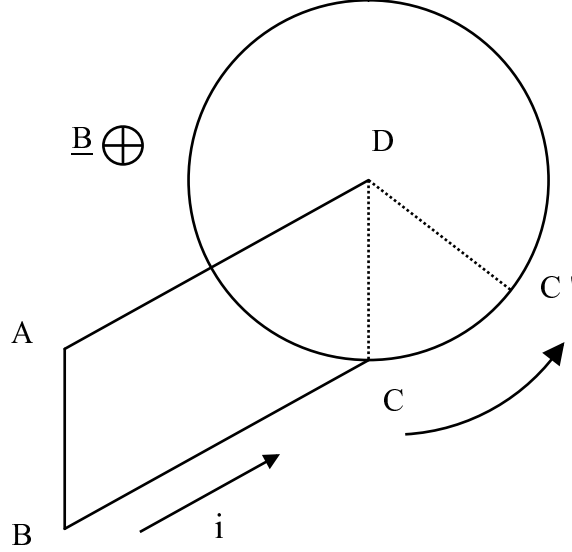


Figure 2: the Faraday disc. A conducting disc rotates with constant angular velocity ω in a uniform and constant magnetic field \vec{B} perpendicular to and entering the plane of the disc. A conducting frame $DABC$ makes conducting contacts with the center and a point on the periphery of the disc. A steady current i circulates as indicated by the arrow.

3.2.2 The Faraday disc: 1

A conducting disc is rotating with constant angular velocity ω in a uniform and constant magnetic field \vec{B} perpendicular to the disc and entering the page (fig. 2). A conducting frame $DABC$ makes conducting contacts with the center and a point on the periphery of the disc. A steady current i circulates as indicated by the arrow.

By applying the general laws (16) or (15) to the integration line at rest $ABCD$, we see that, if we neglect the contribution of the drift velocity:

- since the magnetic field does not depend on time, the first integral in (16) is zero
- equivalently, the sum of the first two integrals in (15) is zero. In the present case, each of the two terms is zero: the flux associated with the circuit does not change and \vec{v}_{line} is zero everywhere, since the integration line has been chosen at rest

- the only surviving term is the one due to the magnetic component of Lorentz force and its value is given by (we are taking the counterclockwise direction for the line integration):

$$\mathcal{E} = \frac{1}{2}\omega BR^2 \quad (20)$$

where R is the disc radius. In doing this calculation, we have neglected - as explained at the end of section 3.1, the charge drift velocity (we shall take it into account in section 3.2.7).

Of course, the *emf* given by equation (20) is induced along any radius, as it can be seen by considering the circuit $ABCC'DA$. If the radius $C'D$ is considered at rest, the case is the same as that of the circuit $ABCD$ discussed just before. If the radius $C'D$ is considered in motion with angular velocity ω , we see again, both from equation (16) and equation (15), that the only surviving term is the one due to the magnetic component of Lorentz force with the only difference that, now, the first term of equation (15) $(1/2)\omega BR^2$ is cancelled out by the second $-(1/2)\omega BR^2$, due to the movement of the line elements of the radius DC' ; the line elements lying on the CC' arc give a null contribution.

Let us now see how the flux rule deals with the Faraday disc. If we choose the lines *at rest* $ABCD$ or $ABCC'DA$, we find that the induced *emf* is zero; if we choose the integration line $ABCC'DA$, with the radius $C'D$ *considered in motion*, we obtain the correct result $\mathcal{E} = (1/2)\omega BR^2$.

As in the case of the moving bar, the flux rule gets the correct result only because the two line integrals under square brackets in equation (15) cancel each other. As in the case of the moving bar, the physics embedded in the general equation forbids an interpretation of the mathematical result in terms of flux variation: again the physical origin of the induced *emf* is due to the intermediacy of the magnetic component of Lorentz force.

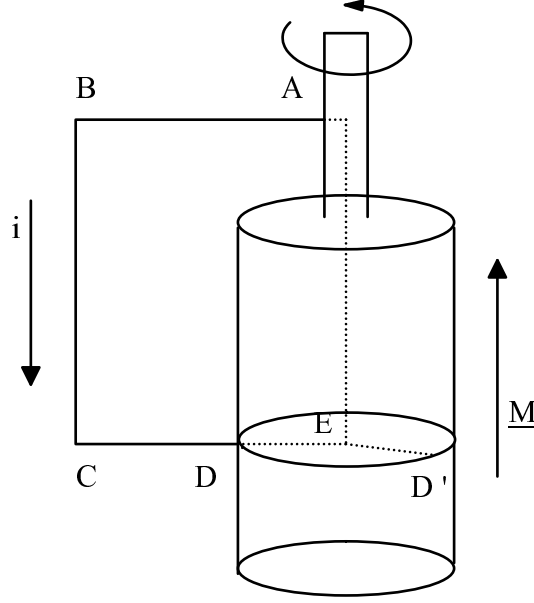


Figure 3: the unipolar induction. A conducting cylindrical magnet rotates about its axis in a counterclockwise direction. A conducting frame $ABCD$ makes conducting contacts between a point on the axis of the magnet and a point on its surface. A steady current i circulates as indicated by the arrow.

3.2.3 The unipolar induction

The so called unipolar induction is illustrated in fig. 3. When the cylindrical and conducting magnet M rotates about its axis with angular velocity ω in the counterclockwise direction, a current flows in the circuit as indicated by the arrow. It is easy to see that the discussion of the unipolar induction can be reduced to that of the Faraday disc for both general law and flux rule: the general law applied to the circuit $ABCDEA$ at rest yields an emf induced in the radius DE given by $(1/2)\omega B \overline{DE}^2$, while the flux rule applied to the same circuit predicts a zero emf . If one consider instead the integration line $ABCDD'EA$ with the radius $D'E$ in motion, analogous to the path $ABCC'DA$ of fig. 2 concerning the Faraday disc, one can follow the same arguments developed in that case.

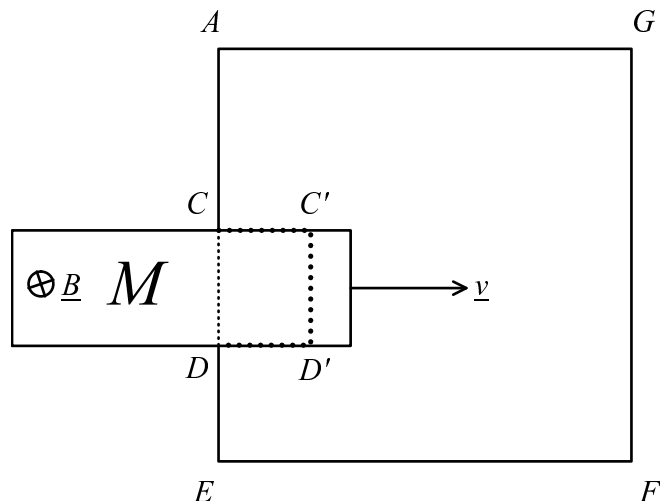


Figure 4: Kaempffer's example. The circuit $DEFGAC$ is closed by the conducting magnet M that is moving to the right with constant velocity \vec{v} . The magnet is sufficiently long in the direction perpendicular to the page so that there is a magnetic field only within the magnet. There is no induced emf in spite of the fact that there is a flux change through the circuit at rest $ACDEFGA$.

3.2.4 The flux varies, but...

In fig. 4 a case discussed by Scanlon, Henriksen and Allen⁹ and originally due to Kaempffer¹⁰ is presented. When the conducting magnet M moves, as indicated, with constant velocity \vec{v} , there is no induced emf - if the magnet is sufficiently long in the direction perpendicular to the page so that there is a magnetic field only within the magnet. If we consider the integration line $ACDEFGA$ at rest, there is a flux variation without induced emf . Also in this case, we must choose *ad hoc* the integration line - for instance the line $ACC'D'DEFGA$, where the segment $C'D'$ moves with the magnet - if we want to save the flux rule. On the other hand, the general law works well, whatever integration line is chosen.

⁹ P.J. Scanlon, R.N. Henriksen and J.R. Allen, "Approaches to electromagnetic induction", Am. Jour. Phys., **37**, pp. 705 - 706 (1969).

¹⁰ F. A. Kaempffer, *Elements of Physics* (Blaisdell Publ. Co., Waltham, Mass., 1967), p. 164; quoted by Scanlon, Henriksen and Allen.

3.2.5 The ‘rocking plates’

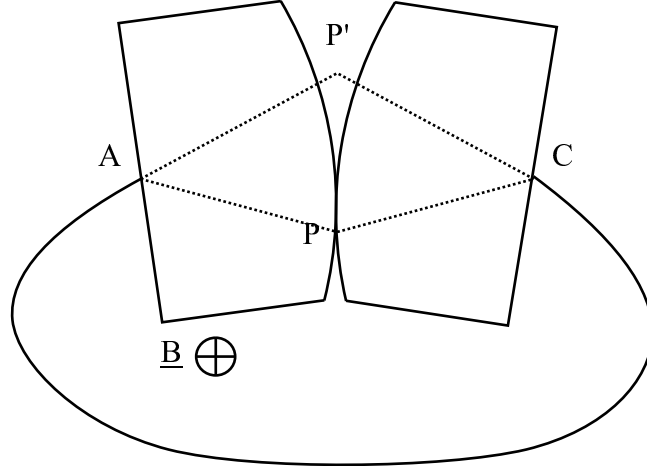


Figure 5: the ‘rocking plates’. Two conducting plates oscillate back and forth in a uniform and constant magnetic field perpendicular to the plates.

This is one of the two ‘exceptions to the flux rule’ discussed by Feynman, Leighton and Sands (fig. 5).¹¹ The two plates oscillate slowly back and forth so that their point of contact moves from P to P' and viceversa. The circuit is closed by a wire that connects point A and C . The magnetic field \vec{B} is perpendicular to the plates and enters the page. The authors write:

If we imagine the “circuit” to be completed through the plates on the dotted line shown in the figure, the magnetic flux through this circuit changes by a large amount as the plates are rocked back and forth. Yet the rocking can be done with small motions, so that $\vec{v} \times \vec{B}$ is very small and there is practically no emf.

According to the general law (15) the sum of the first two terms of equation (15) must be zero. Hence:

$$\oint_l (\vec{v}_{line} \times \vec{B}) \cdot d\vec{l} = -\frac{d}{dt} \int_S \vec{B} \cdot \hat{n} dS \quad (21)$$

¹¹ R. Feynman, R. Leighton and M. Sands *The Feynman Lectures on Physics*, (Addison Wesley, Reading, Ma., 1964), pp. 17 - 3.

Therefore, the third term of equation (15) coming from the magnetic component of Lorentz force equals the magnetic flux variation changed in sign, if, of course, we assume that the charge velocity can be taken equal to the velocity of the element of conductor that contains it (the drift velocity is neglected). The conclusion is that there is an induced *emf*: its average value is given by $-\Delta\phi/\Delta t$ where $\Delta\phi$ is the flux variation between the two extreme positions of the plates and Δt the interval of time taken in going from one position to the other: the induced *emf* gets smaller and smaller as Δt gets larger and larger; when the motion of the plates is very slow, we can conclude with Feynman that ‘there is practically no *emf*’. Notice that the induced *emf* changes in sign (and the current its direction) when the the rocking motion is reversed.

3.2.6 The Corbino disc

The discussion of this case will show how the charge drift velocity, always neglected before, plays its role in the building up of the induced electromotive field and, *therefore*, of the induced *emf*. In 1911, Corbino studied theoretically and experimentally the case of a conducting disc with a hole at its center. If a voltage is applied between the inner and the outer periphery of the disc, a radial current will flow, provided that the experimental setup is realized in a way suitable for maintaining the circular symmetry: the inner and outer periphery are covered by highly conducting electrodes; therefore, the inner and outer periphery are two equipotential lines. If a uniform and constant magnetic field is applied perpendicularly to the disc, a circular current will flow in the disc.¹²

¹² O.M. Corbino, “Azioni elettromagnetiche dovute agli ioni dei metalli deviati dalla traiettoria normale per effetto di un campo”, *Il Nuovo Cimento* **1**, 397 - 419 (1911). A german translation of this paper appeared in *Phys. Zeits.*, **12**, 561 - 568 (1911). For a historical reconstruction see: S. Galdabini and G. Giuliani, “Magnetic field effects and dualistic theory of metallic conduction in Italy (1911 - 1926): cultural heritage, epistemological beliefs, and national scientific community”, *Ann. Science* **48**, 21 - 37 (1991). As pointed out by von Klitzing, the quantum Hall effect may be considered as an ideal (and quantized) version of the Corbino effect corresponding to the case in which the current in the disc, with an applied radial voltage, is only circular: K. von Klitzing, “The ideal Corbino effect”, in: P.E. Giua ed., *Commemorazione di Orso Mario Corbino*, (Centro

The first theoretical treatment of this case is due, as far as I know, to Boltzmann who wrote down the equations of motion of charges in combined electric and magnetic fields.¹³ Corbino, apparently not aware of this fact, obtained the same equations already developed by Boltzmann. However, while Boltzmann focused on magnetoresistance effects, Corbino interpreted the theoretical results in terms of radial and circular currents and studied experimentally the magnetic effects due to the latter ones.

The application of the general law of electromagnetic induction to this case leads to the same results usually obtained (as Boltzmann and Corbino did) by writing down and solving the equations of motion of the charges in an electromagnetic field (by taking into account, explicitly or implicitly, the scattering processes).

If I_{radial} is the radial current, the radial current density $J(r)$ will be:

$$J(r) = \frac{I_{radial}}{2\pi r s} \quad (22)$$

and the radial drift velocity:

$$v(r)_{drift} = \frac{I_{radial}}{2\pi r s n e} \quad (23)$$

where s is the thickness of the disc, n the electron concentration and e the electron charge. In the present case the general law of electromagnetic induction assumes the simple form of equation (8) with $v_{charge} = v_{drift}$; therefore, the induced *emf* around a circle of radius r is given by:

$$\mathcal{E}_{circular} = \oint_0^{2\pi r} (\vec{v}(r)_{drift} \times \vec{B}) \cdot \vec{dr} = \frac{I_{radial} B}{s n e} \quad (24)$$

The circular current $dI(r)_{circular}$ flowing in a circular strip of radius r and section $s \cdot dr$ will be, if ρ is the resistivity:

$$dI_{circular} = \frac{\mathcal{E}_{circular} s dr}{\rho 2\pi r} = \frac{\mu B}{2\pi} I_{radial} \frac{dr}{r} \quad (25)$$

Stampa De Vittoria, Roma, 1987), pp. 43 - 58.

¹³ L. Boltzmann, *Anzeiger der Kaiserlichen Akademie der Wissenschaften in Wien*, **23**, (1886), 77 - 80; *Phil. Mag.*, **22**, (1886), 226 - 228.

and the total circular current:

$$I_{circular} = \frac{\mu B}{2\pi} I_{radial} \ln \frac{r_2}{r_1} \quad (26)$$

where μ is the electron mobility, r_1 and r_2 the inner and outer radius of the disc (we have used the relation $\mu = 1/\rho ne$). Equation (26) is the same as that derived and experimentally tested by Corbino.

The power dissipated in the disc is:

$$W = (I^2 R)_{radial} + (I^2 R)_{circular} = I_{radial}^2 R_{radial} (1 + \mu^2 B^2) \quad (27)$$

where we have used equation (26) and the two relations:

$$R_{radial} = \frac{\rho}{2\pi s} \ln \frac{r_2}{r_1} \quad (28)$$

$$R_{circular} = \frac{\rho^2}{s^2} \frac{1}{R_{radial}} \quad (29)$$

Equation (27) shows that the phenomenon may be described as due to an increased resistance $R_{radial}(1 + \mu^2 B^2)$: this is the magnetoresistance effect. The circular induced *emf* is ‘distributed’ homogeneously along each circle. Each circular strip of section $s \cdot dr$ acts as a battery that produces current in its own resistance: therefore, the potential difference between two points arbitrarily chosen on a circle is zero. Hence, as it must be, each circle is an equipotential line.

The above application of the general law yields a description of Corbino disc that combine Boltzmann (magnetoresistance effects) and Corbino (circular currents) point of view and shows how the general law can be applied to phenomena traditionally considered outside the phenomenological domain for which it has been derived.

3.2.7 The Faraday disc: 2

The discussion of Corbino disc helps us in better understanding the physics of the Faraday disc. Let us consider a Faraday disc in which the circular symmetry is

conserved. This may be difficult to realize; anyway, it is interesting to discuss it. As shown above, the steady condition will be characterized by the flow of a radial and of a circular current. The mechanical power needed to keep the disc rotating with constant angular velocity ω is equal to the work per unit time done by the magnetic field on the rotating radial currents. Then, it will be given by:

$$W = \int_0^{2\pi} \int_{r_1}^{r_2} (J_{radial} r d\alpha s)(B dr)(\omega r) = I_{radial} \frac{1}{2} \omega B (r_2^2 - r_1^2) \quad (30)$$

where the symbols are the same as those used in the previous section. The point is that the term

$$\mathcal{E} = \frac{1}{2} \omega B (r_2^2 - r_1^2) \quad (31)$$

is the induced *emf* due only to the motion of the disc. This *emf* is the primary source of the induced currents, radial and circular. Therefore, the physics of the Faraday disc with circular symmetry, may be summarized as follows:

- a) the source of the induced currents is the induced *emf* due to the rotation of the disc
- b) the primary product of the induced *emf* is a radial current
- c) the drift velocity of the radial current produces in turn a circular induced *emf* that give rise to the circular current

4 General law and flux rule

The cases discussed above must be considered as illustrative examples. Obviously, the conceptual foundations of the general law and of the flux rule can be discussed without any reference to particular cases. The general law has been already dealt with in great detail. Let us now focus our attention on the flux rule.

Let us define - this, of course, is not our choice - the induced *emf* as

$$\mathcal{E} = \oint_l \vec{E} \cdot d\vec{l} \quad (32)$$

where \vec{E} obeys Maxwell equations. As shown above (equations 5 - 6), this leads immediately to the conclusion that the induced *emf* is given by:

$$\mathcal{E} = \oint_l \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \int_S \vec{B} \cdot \hat{n} dS - \oint_l (\vec{v}_{line} \times \vec{B}) \cdot d\vec{l} \quad (33)$$

or, in terms of the vector potential, by (equations 10 - 11):

$$\mathcal{E} = \oint_l \vec{E} \cdot d\vec{l} = -\oint_l \frac{\partial \vec{A}}{\partial t} \cdot d\vec{l} \quad (34)$$

and is always zero when $\partial \vec{B} / \partial t = 0$.

Of course, this result is not satisfactory, if we want to save the flux rule. Therefore, one may suggest to define the induced *emf* as:

$$\mathcal{E} = \oint_l \vec{E} \cdot d\vec{l} + \oint_l (\vec{v}_{line} \times \vec{B}) \cdot d\vec{l} = -\frac{d}{dt} \int_S \vec{B} \cdot \hat{n} dS \quad (35)$$

thus reestablishing the flux rule. This is, for instance, the choice made by Scanlon, Henriksen and Allen.¹⁴ However:

- since the velocity appearing in equation (35) is clearly the velocity of the line element $d\vec{l}$, this definition has no physical meaning.
- the definition is *ad hoc*. As a consequence, as in the paper by Scanlon, Henriksen and Allen, the line of integration must be chosen *ad hoc* for obtaining predictions in accordance with the experimental findings.

5 Conclusions

The straightforward application of two fundamental laws of electromagnetism – Maxwell equation $\text{rot} \vec{E} = -\partial \vec{B} / \partial t$ and Lorentz law $\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$ – leads to a general law for electromagnetic induction phenomena: it includes, as a particular case, the standard flux rule. The treatment given in this paper shows that standard derivations:

¹⁴ P.J. Scanlon, R.N. Henriksen and J.R. Allen, “Approaches to electromagnetic induction”, Am. Jour. Phys., **37**, (1969), p. 701.

- a) Fail to recognize that the velocity involved in the integration of Maxwell equation (2) is the velocity of the line elements composing the closed path of integration and *not* the velocity of the charges.
- b) Overlook the fact that the expression of Lorentz force constitutes a fundamental postulate not included in Maxwell equations; this postulate not only gives physical meaning to the fields solutions of Maxwell equations – when applied to charges – but allows also the prediction of new phenomena.
- c) In relevant cases, when a part of the circuit is moving, they are misleading as far as the physical origin of the phenomenon is concerned: they attribute the effect to a flux variation when, instead, the origin lies in the intermediacy of the magnetic component of Lorentz force.

Finally, the present derivation:

- provides with a rigorous mathematical treatment the statement according to which electromagnetic induction phenomena are the product of two independent processes: time variation of magnetic field and effects of magnetic field on moving charges.
- yields a law that can be successfully applied to phenomena - those presented by Corbino disc - till now considered outside the phenomenological domain of electromagnetic induction.

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