MOTIONS AND REST IN GENERAL RELATIVITY
A HISTORICAL-CRITICAL NOTE

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Abstract. No class of privileged coordinate-systems (as, *e.g.*, the Lorentzian frames of special theory) exists in general relativity. This fact has many momentous consequences, in particular with regard to the concepts of motion and rest. I give here a rather detailed treatment of this subject.

Summary – **1**. In general relativity (GR) the concept of coordinate system has been "erweicht" (Weyl) – *i.e.*, "mollified". As a consequence, the concepts of motion and rest have undergone a radical relativization. – **2**. In GR a coordinate transformation must be one-to-one and continuous (Hilbert, Eddington). Consequences regarding some celebrated transformations of coordinates. – **3**. Kretschmann-Mie and Fock *versus* the Founding Fathers of GR; the vain search for a general-relativistic class of privileged frames. – **4**. **4bis**. No "mechanism" exists for the generation of undulatory gravitational fields. – **5**. A suggestive similitude. – *Appendix*: On Killing equations. –

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1. – At p.268 of Weyl's *Raum-Zeit-Materie* we read [1]:

"... *der Begriff der Relativbewegung zweier Körper gegeneinander in der allgemeinen Relativitätstheorie ebensowenig einen Sinn hat wie der Begriff der absoluten Bewegung eines einzigen.* Solange man noch den starren Bezugskörper zur Verfügung hatte und zu der Objektivität der Gleichzeitigkeit glauben konnte, auf dem Standpunkte Mach's etwa, unter der Herrschaft der kinematischen Gruppe gab es eine relative Bewegung; aber in der allgemeinen Relativitätstheorie hat sich das Koordinatensystem so "erweicht", daß auch davon nicht mehr die Rede sein kann. Wie die beiden Körper sich auch bewegen mögen, immer kann ich durch Einführung eines geeigneten Koordinatensystems die beide zusammen auf Ruhe transformieren." Accordingly, we see that in general relativity the concepts of motion and rest have undergone a radical relativization with respect to the analogous concepts of special relativity: not only any motion of any body has lost any absolute nature, but even any relative motion of two bodies can be transformed into rest. (In particular, the acceleration has lost its privileged character that it enjoyed in the special theory).

Of course, in the choice of a reference system it is suitable – for *practical* reasons – to avoid the frames with "inertial and transport forces". Thus, to compute the gravitational field of a celestial body, we shall assume that it is very far from other masses. In his two fundamental memoirs, Schwarzschild computed the gravitational field of a material point (at rest) [2], and the gravitational field of a homogeneous sphere of an incompressible fluid (at rest) [3]. The general expression of the interval ds of Schwarzschild manifold generated by a material point of mass m was derived *ex-novo* by Levi-Civita, following a detailed, *explicitly geometrical* approach [4]. The result ($c = G = 1$):

(1)
$$\mathrm{d}s^2 = \left(1 - \frac{2m}{R(r)}\right) \mathrm{d}t^2 - \left(1 - \frac{2m}{R(r)}\right)^{-1} [\mathrm{d}R(r)]^2 - [R(r)]^2 \left(\mathrm{d}\vartheta^2 + \sin^2\vartheta \mathrm{d}\varphi^2\right) ,$$

where $R(r)$ is any regular function of r, such that the metric becomes Minkowskian at $r = \infty$. Eq.(1) holds for $R(r) > 2m$, because for $R(r) \leq 2m$ the metric loses its essential pseudo-Riemannian character. In particular, the surface area $4\pi(2m)^2$ of the object for which $R(r) = 2m$ represents a non-existent (invariant) notion. However, if we adopt the *original* Schwarzschild's choice $R(r) = \left[r^3 + (2m)^3\right]^{1/3}$, or Brillouin's choice [5] $R(r) = r+2m$, we get a *maximally extended* manifold with a *unique* singular point at $r = 0$.

2. – At p. 225 of his treatise [6] Eddington wrote: "... the arbitrariness of the coordinate system is limited. We may apply any continuous transformation; but our theory does not contemplate a discontinuous transformation of coordinates, such as would correspond to a re-shuffling of the points of the continuum. There is something corresponding to an *order* of *enumeration* of the points which we desire to preserve, when we limit the changes of coordinates to continuous transformations." And in his memoir *Die Grundlagen der Physik* [7] Hilbert specified: "... nenne ich eine Maßbestimmung oder ein Gravitationsfeld $g_{\mu\nu}$ an einer Stelle *regulär*, wenn es möglich ist, durch umkehrbar eindentige Transformation ein solches Koordinatensystem einzuführen, daß für dieses die entsprechenden Funktionen $g'_{\mu\nu}$ an jener Stelle regulär, d.h. in ihr und in ihrer Umgebung stetig und beliebig oft differenzierbar sind und eine von Null verschiedene Determinante g' haben." Now, these conditions are disregarded in the baroque procedure by Kruskal-Szekeres [8], which uses a coordinate transformation whose derivatives are singular at $R(r) \equiv r = 2m$ in such a way to give a metric that is regular for $0 < r \leq 2m$; further, this new metric *duplicates* the standard metric $R(r) \equiv r$.

The future historians of physics will explain the reasons (if any) for which so many physicists have overlooked the *fact* that Schwarzschild's original solution is *maximally extended* and *perfectly satisfying* from both the mathematical and physical standpoints.

3. – Kretschmann [9], Mie [10] and Fock [11] declared that the locution "general relativity" is a nonsense, because Einstein's theory does not admit a subclass of physically privileged coordinate-systems as the Lorentzian systems of special relativity, for which all physical phenomena have an identical course. Let us consider, on the contrary, an observer on the top of a belfry and a voyager in a train which undergoes a strong deceleration. Observer and voyager are on the same footing from the *kinematical* standpoint, but *dynamically* things stand otherwise: the voyager undergoes a backwards violent shove, while the observer does not feel any sensation. Now, the above authors affirm that two reference frames are physically equivalent only if the kinematical and the dynamical aspects of any phenomenon are in accord. From the mathematical point of view, this conviction rests on the geometric concepts of invariance and covariance as developed by Klein and by Cartan.

With reference to the papers by Kretschmann and by Mie (Fock's writings on the problem belong only to the Fifty Years of past century), the question was admirably clarified by Pauli [12], who concluded in favour of the opposite conviction of the Founding Fathers of general relativity: according to them, the "inertial and transport forces" are not physically important, because they are only "seeming" forces. Accordingly, the locution "general relativity" is perfectly appropriate; of course, in it the substantive "relativity" has a different meaning with respect to the "relativity" of special theory.

Fock affirmed to have discovered a subclass of coordinate systems of Einstein's gravitational theory, which correspond to the Lorentzian coordinate systems of the special relativity: this subclass would be composed of the well known *harmonic* systems. If $\Psi(y^0, y^1, y^1, y^3)$ is a scalar field, its d'Alembertian $\Box\Psi(y)$ is given by

$$\Box\,\Psi = \frac{1}{\sqrt{-g}}\,\frac{\partial}{\partial y^j}\left(g^{jk}\sqrt{-g}\,\frac{\partial\Psi}{\partial y^k}\right) \quad , \quad (j,k=0,1,2,3) \quad ; \tag{2}$$

the four *harmonic conditions*

$$\frac{\partial}{\partial y^j}\left(g^{jk}\sqrt{-g}\right) = 0 \quad , \tag{3}$$

give

$$\Box\, y^k = \frac{1}{\sqrt{-g}}\,\frac{\partial}{\partial y^j}\left(g^{jk}\sqrt{-g}\right) = 0 \quad ; \tag{4}$$

we see that each of the functions $\Psi = y^0$; $\Psi = y^1$; $\Psi = y^2$; $\Psi = y^3$ is a solution of d'Alembert equation $\Box\,\Psi = 0$. (Remark that the harmonic coordinate y^m, $(m = 0, 1, 2, 3)$, is here formally considered as a scalar field.)

To demonstrate his statement, Fock tried to prove that every harmonic system which describes an insulated material distribution can be transformed *via* a Lorentzian change of coordinates into another given harmonic system. His proof, however, is not convincing, because it makes use of *ad hoc* hypotheses. In reality, the harmonic coordinates represent only a suitable reference frame for the solution of several problems of Einstein's gravitational theory, General relativity does not admit any subclass of privileged frames.

4. – As it is emphasized by Weyl (see the quotation in sect. **1**), every moving body can be reduced to rest through a suitable change of spacetime coordinates. Further, even two bodies in a relative motion can be reduced to rest by choosing a convenient reference system. (Under this respect, general relativity is radically different from Maxwell theory).

An immediate logical consequence is that there is *no* "mechanism" for the generation of gravitational waves; in particular, a binary star system does not send forth any gravity undulation.

The *physical* non-existence of gravitational waves can be proved in several ways. For instance:

i) As it was remarked by Levi-Civita [13], an undulatory solution of Einstein equations $R_{jk} = 0$ does not have a *true* energy tensor –

ii) The wave nature of a given g_{jk} can be always *destroyed* by an appropriate change of coordinates –

iii) As a consequence of Einstein field equations, the gravitational motions of the particles of a "cloud of dust" are described by *geodesic* lines [14]. A result that can be extended to the general case in which the above particles interact also with *other* fields, different from the gravitational one [15].

iv) An intuitive argument. Assume that in an given motion ("natural", or "externally" impressed by a spring, *e.g.*) of a point-mass, a gravitational wave is emitted in the time interval from t to $t + \Delta t$. Now, the same *kinematical* properties of this motion (velocity, acceleration, time derivative of the acceleration, *etc.*), in an equal time interval Δt, can be reproduced in a motion of the same body under the action of "external" *gravitational* forces of a suitable intensity. But in this case we have a *geodesic* motion. The conclusion is clear.

4bis. – The above statement concerning the binary systems can be also easily proved with a straightforward computation. Remark that all the conventional calculations regarding the revolutions of the famous BPSR1913+16 make use of the *linear* approximation of general relativity. Now, these computations disregard the following fundamental *fact*, which proves their full nonsense: in the linear approximation, the equations $\partial T^{jk}/\partial x^k = 0$, when $T^{jk} = \varrho\,(\mathrm{d}x^j/\mathrm{d}s)\,(\mathrm{d}x^k/\mathrm{d}s)$ – where $\varrho\,\mathrm{d}x^j/\mathrm{d}s$ gives the density and flow of the matter –, have as a consequence that the motions of the particles are represented by *rectilinear* ***geodesics***; the conclusion is obvious. More generally, if we apply the *exact* relativity, the covariant equations $T^{jk}_{:k} = 0$ – where the colon denotes covariant derivative – tell us that the particles follow *curvilinear* ***geodesics***. The conclusion is clearly the same.

As it is well known to many observational astrophysicists, there are realistic explanations of the time decrease of the revolution period of BPSR1913-+16 [16].

5. – *A suggestive similitude.* Marc Bloch (1886-1944), an acute historian, wrote in 1924 a documented book on *Les rois thaumaturges – Étude sur le caractère surnatural attribué à la puissance royale particulièrement en France et en Angleterre.* During almost thousand years, the European people believed that the kings of France and of England had the faculty to cure the scrofulouses by imposition of hands. This belief was favoured by the circumstance that the scrofula has many mutable manifestations, with seeming recoveries. The existence of a magic power of the royal thaumaturgists was believed by the kings themselves.

Today's convictions of the physical existence of wonderful "globes" with surface area $4\pi(2m)^2$, and of undulatory gravitational fields, with some extraordinary properties, have a clear resemblance to the old beliefs in a royal thaumaturgy. Let us hope that the beliefs in the BH's (never observed, in reality) and in the GW's (never experimentally detected) will last less than a millennium.

A discussion with my friend Dr. S. Antoci, a strenuous supporter of Kretschmann's standpoint, is gratefully acknowledged.

Appendix: On Killing equations

"... a Riemannian space admits a group of transformations into itself, when each transformation leaves invariant the metric properties of the space." With this sentence, Eisenhart begins his treatment of the famous Killing equations (1892), and of their geometrical consequences [17]. Here I limit myself to the considerations of pp. 208-209 (sect. **51**) of [17].

If a manifold V_n characterized by

(A.1) $$\mathrm{d}s^2 = g_{jk}(x)\,\mathrm{d}x^j\,\mathrm{d}x^k \quad , \quad (j,k=1,2,..,n) \quad ,$$

is subjected to an infinitesimal point-displacement δx^j:

(A.2) $$x^{*j} := x^j + \xi^j(x)\,\delta\,\sigma \quad ,$$

where $\delta\,\sigma$ is an infinitesimal parameter, we have, as it can be easily seen:

(A.3) $$\delta(\mathrm{d}x^j) = \mathrm{d}(\delta x^j) = \frac{\partial \xi^j}{\partial x^k}\,\mathrm{d}x^k\,\delta\,\sigma \quad ,$$

(A.3')
$$\delta\, g_{jk} = \frac{\partial g_{jk}}{\partial x^m}\,\xi^m\,\delta\,\sigma \quad .$$

The necessary and sufficient condition that $\delta(\mathrm{d}s^2)=0$ is clearly:

(A.4)
$$\xi^m\,\frac{\partial g_{jk}}{\partial x^m} + g_{jm}\,\frac{\partial \xi^m}{\partial x^k} + g_{km}\,\frac{\partial \xi^m}{\partial x^j} = 0 \quad ,$$

which can be written

(A.4')
$$\xi_{j:k} + \xi_{k:j} = 0 \quad ,$$

where the colon denotes covariant derivative. Eqs. (A.4') – *Killing equations* – tell us that $(\mathrm{d}s^2)^*$ is only a displaced $\mathrm{d}s^2$.

If V_n admits a coordinate system for which, *e.g.*:

(A.5)
$$\xi^i = \delta^j_1 \quad ,$$

eqs. (A.4) reduce to

(A.6)
$$\frac{\partial g_{jk}}{\partial x^k} = 0 \quad ,$$

i.e., the g_{jk}'s are independent of x^1, and $\mathrm{d}s^2$ remains unaltered by the *finite* displacements

(A.7)
$$x^{*1} = x^1 + \sigma \quad ; \quad x^{*r} = x^r \quad , \quad (r = 2, 3, .., n) \quad ,$$

of the group G_1 generated by (A.2).

We have an instance of a group G_1 in the *static* solutions of Einstein field equations given, *e.g.*, by Schwarzschild manifolds and by the axially symmetric manifolds of Levi-Civita and Weyl, if we identify the parameter σ with the coordinate time t; a physically trivial group G_1, of course. The *static* nature of a given problem of general relativity must be determined by *practical* criteria. –

References

[1] H. Weyl, *Raum-Zeit-Materie*, Siebente Auflage (Springer-Verlag, Berlin, *etc.*) 1988. See also: *Idem*, *Mathematische Analyse des Raumproblems* (Julius Springer, Berlin) 1923, Siebente Vorlesung. – Italian version by A. Loinger, *Analisi matematica del problema dello spazio* (Zanichelli, Bologna) 1991, settima lezione. – See further: A. Loinger, *Riv. Nuovo Cimento*, **11** (no.8) (1988) 1.

[2] K. Schwarzschild, *Berl. Ber.*, (1916) 189; for an English version by S. Antoci and A. Loinger, see: *arXiv:physics/9905030* (May 12th, 1999).

[3] K. Schwarzschild, *Berl. Ber.*, (1916) 424; for an English version by S. Antoci, see: *arXiv:physics/9912033* (December 16th, 1999).

[4] T. Levi-Civita, *The Absolute Differential Calculus (Calculus of Tensors)* – (Dover Publications, Inc., Mineola, N.Y.) 2005, p. 408 *sqq.* (Originally published: London: Blackie and Son, 1926).

[5] M. Brillouin, *Journ. Phys. Rad.*, **23** (1923) 43; for an English version by S. Antoci, see: *arXiv:physics/0002009* (February 3rd, 2000).
[6] A. S. Eddington, *The Mathematical Theory of Relativity*, Second Edition (Cambridge University Press, Cambridge) 1960.
[7] D. Hilbert, *Math. Annalen*, **92** (1924) 1.
[8] M. D. Kruskal, *Phys. Rev.*, **119** (1960) 1743; G. Szekeres, *Publ. Mat. Debrecen*, **7** (1960) 285.
[9] E. Kretschmann, *Ann. Physik*, **53** (1917) 575.
[10] G. Mie, *Ann. Physik*, **62** (1920) 46.
[11] V. Fock, *The Theory of Space, Time and Gravitation*, (Pergamon Press, Oxford, *etc.*), 1964; Introduction, *passim*, sect. **93**.
[12] W. Pauli, *Teoria della Relatività* (Boringhieri, Torino) 1958, sect. **52**.
[13] T. Levi-Civita, *Rend. Acc. Lincei*, **26** (1917) 381; for an English version by S.Antoci and A. Loinger, see *arXiv:physics/9906004* (June 2nd, 1999).
[14] A. Loinger, *arXiv:astro-ph/0003230* (March 16th, 2000).
[15] A. Loinger, *arXiv:physics/0606019* (June 2nd, 2006); *Idem*, *arXiv:0804.3991* [physics.gen-ph] 24 Apr 2008; *Idem*, *arXiv:1006.3844* [physics.gen-ph] 19 Jun 2010.
[16] A. Loinger, *arXiv:physics/0502089* (February 16th, 2005).
[17] L.P. Eisenhart, *Continuous Groups of Transformations* (Dover Publications, Inc., New York) 1961, sects. **51÷58**.

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