ON DE SITTER'S MODEL OF SPINNING SPHERE AND ITS FRAME-DRAGGING EFFECT

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Abstract. In a basic paper of 1916, de Sitter gave, in particular, a treatment of a special case (equatorial geodesics) of the frame-dragging effect, which is generated by a model of spinning sphere. We prove in the present Note that this model is inconsistent with the adopted *linear* approximation of general relativity.

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1. – In the *linear* approximation of general relativity (GR) the mass tensor $T^{\alpha\beta}$, $(\alpha, \beta = 1, 2, 3, 4)$ of a perfect fluid can be written as follows (CGS system of units), cf. [2]:

$$c^2\, T^{\alpha\beta} = \left(\mu + \frac{p}{c^2}\right) \frac{\mathrm{d}y^\alpha}{\mathrm{d}\tau} \frac{\mathrm{d}y^\beta}{\mathrm{d}\tau} - \eta^{\alpha\beta} p \quad , \tag{1}$$

where: μ is the rest-mass density, which includes the mass corresponding to the compressional potential energy; p is the pressure; y^1, y^2, y^3 are Cartesian rectangular coordinates, $y^4 = ct$; $\mathrm{d}\tau = \mathrm{d}s/c$; $\eta^{\alpha\beta}$ is the customary Minkowskian tensor ($\eta^{44} = 1$). We have:

$$\mu = \varrho \left(1 + \frac{\Pi}{c^2}\right) \quad ; \quad \mu = \mu(p) \quad ; \quad \varrho = \varrho(p) \quad , \tag{2}$$

if ϱ is the density of that part of the rest-mass which does not change in the motion, and

$$\Pi = \Pi(\varrho, p) := \int_0^p \frac{\mathrm{d}p}{\varrho} - \frac{p}{\varrho} \quad . \tag{3}$$

From the conservation equations

$$\frac{\partial T^{\alpha\beta}}{\partial y^\beta} = 0 \quad . \tag{4}$$

which follow from the field equations, we get the equations of motion of the particles of the fluid:

$$\left[\varrho + \frac{1}{c^2}(\varrho\Pi + p)\right] \frac{\mathrm{d}u^\alpha}{\mathrm{d}\tau} = \eta^{\alpha\beta} \frac{\partial p}{\partial y^\beta} - \frac{1}{c^2} \frac{\mathrm{d}p}{\mathrm{d}\tau} u^\alpha \quad , \tag{5}$$

if $u^\alpha = \mathrm{d}y^\alpha/\mathrm{d}\tau$. The fourth of eqs. (5) can be substituted by the continuity equation

$$\frac{\partial}{\partial y^\alpha}(\varrho\, u^\alpha) = 0 \quad . \tag{6}$$

We see from (5) *that velocity and acceleration of the particles cannot be prescribed* a priori: i*n the linear approximation of GR there is no gravitational contribution to the pressure p, and therefore a gaseous body tends to dissolve.*

2. – In paper [1] de Sitter wrote the hydrodynamical mass tensor $T^{\alpha\beta}$ in the following system of polar coordinates:

$$y^1 = r\cos\varphi\cos\vartheta \quad ; \quad y^2 = r\cos\varphi\sin\vartheta \quad ; \quad y^3 = r\sin\varphi \quad , \tag{7}$$

from which:

$$\mathrm{d}s^2 = -\mathrm{d}r^2 - r^2\cos^2\varphi\,\mathrm{d}\vartheta^2 - r^2\mathrm{d}\varphi^2 + c^2\mathrm{d}t^2 \quad . \tag{8}$$

Putting $c = 1$, if $\xi^1 \equiv r$; $\xi^2 \equiv \vartheta$; $\xi^3 \equiv \varphi$; $\xi^4 = t$ – and $\overset{0}{g}_{11} = -1$; $\overset{0}{g}_{22} = -r^2\cos^2\varphi$; $\overset{0}{g}_{33} = -r^2$; $\overset{0}{g}_{44} = 1$, we get

$$T_{\gamma\delta} = \overset{0}{g}_{\alpha\gamma}\,\overset{0}{g}_{\beta\delta}\,\frac{\mathrm{d}\xi^\alpha}{\mathrm{d}s}\frac{\mathrm{d}\xi^\beta}{\mathrm{d}s}\,[\varrho(1+\Pi)+p] - p\,\overset{0}{g}_{\gamma\delta} \quad ; \tag{9}$$

de Sitter's spinning sphere is characterized by $\mathrm{d}r/\mathrm{d}s = 0 = \mathrm{d}\varphi/\mathrm{d}s$, and by $\mathrm{d}\vartheta/\mathrm{d}t = \omega$, with the approximations $\mathrm{d}s \approx \mathrm{d}t$, $\omega^2 \approx 0$, *i.e.* neglecting the quadratic expressions of the three-velocities. The component T_{24}, given by

$$T_{24} \approx r^2\cos^2\varphi\,\omega\,\varrho \quad , \tag{10}$$

has obviously a particular role. Starting from (10), de Sitter found the gravitational field generated by the spinning sphere, *and investigated its effect on the light-rays and the test-particles which describe equatorial geodesics. His results coincide essentially with those of Lense and Thirring* [3].

However, the base of de Sitter's method is questionable, as it can be seen in a plain way by considering the Cartesian form (1) of the mass tensor, and its consequence (5). As we have emphasized, we are not allowed to impose *ad libitum* conditions on the motions of the particles of the sphere; in the *exact* GR, and in its *linearized* version, the equations of motion of matter are a mathematical consequence of the field equations. (But this became clear after 1927, while de Sitter wrote in 1916; at that date, only the Hilbertian formulation of GR (1915) attested *en passant* the above fact).

References

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